Introduction to Burmester Theory

(As told to me by Ludwig Burmester himself in der olde Munich Wursthaus in der shpring of 1876, but mit graphical und complex number haditions since dot time by odders und mineself)

Roger Kaufman
The Mechanisms Man

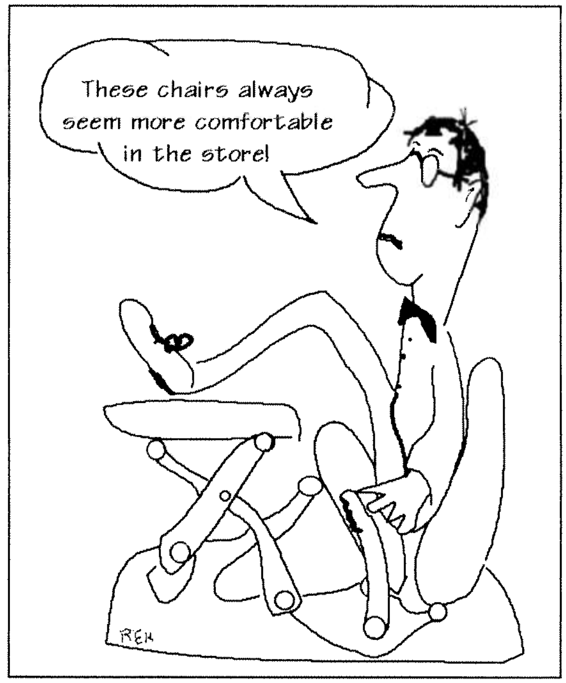
Kaufman Carrot Linkage

Stare at picture and move close to nose
to animate linkage

This book is in loving memory of two of the
inspiring teachers who shaped my life:
my father, Hyman Mendel Kaufman
and George N. Sandor.

In designing linkage mechanisms, a major problem is figuring out where to put pivots or sliders so as to create a desired mechanical motion. As a designer you generally have a pretty good idea of how you want a mechanism to move but you don't have the foggiest idea of how to achieve that motion.

For instance, if you are designing a reclining chair, you probably know how you want the footrest to move but you don't know what sort of a linkage will make that happen. Similarly, if you are designing the wing flap of an airplane, you probably know what it must look like in the open and stowed positions but you don't know what sort of mechanism will accomplish that task and still fit within the confines of the wing.

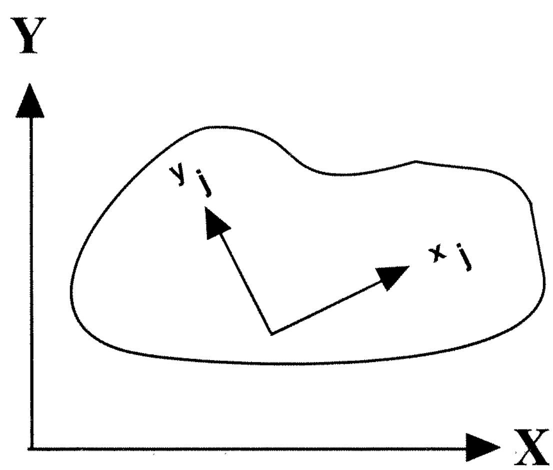


One of the problems you face as a designer is that you have too much confusing information around you so it is hard to figure out what really is important in terms of influencing the kinematic performance of the mechanism and what is just a distraction. From a mechanism design point of view, the problem of designing

a wing flap is just like that of designing a reclining chair—you need to figure out such things as how many links to use and where to put the pivots or sliders.

In one case you are limited because the pivots can't poke into the tush of the person sitting in the chair. In the other case you are limited by the fact that the parts of the wing flap must lie within the wing of the aircraft whose development is being funded and not within the wings of planes belonging to the bankrupt airline in the next hanger.

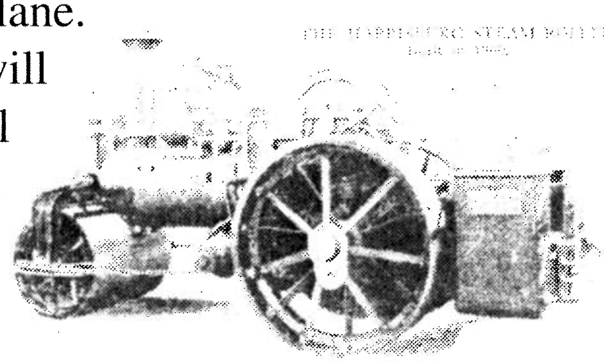
In the pages to follow, we will develop general mathematical methods for designing mechanisms which will work on all kinds of problems. You will learn to look at the motions involved abstractly and not clouded by non-kinematic considerations. We will extract the kinematic essence of the problem. The kinematic essence is the same regardless of whether you are designing a garlic processing machine, a manure spreader, a water saving toilet, or a machine to package baby diapers.



For the purpose of this weighty tome, the term "plane" will be used to mean a rigid body of some arbitrary shape and size, whose motion is always planar. By that I mean you can select some fixed X Y Z reference coordinate system and the body will only be allowed to move by translating in the X or Y directions or by rotating about the Z axis.

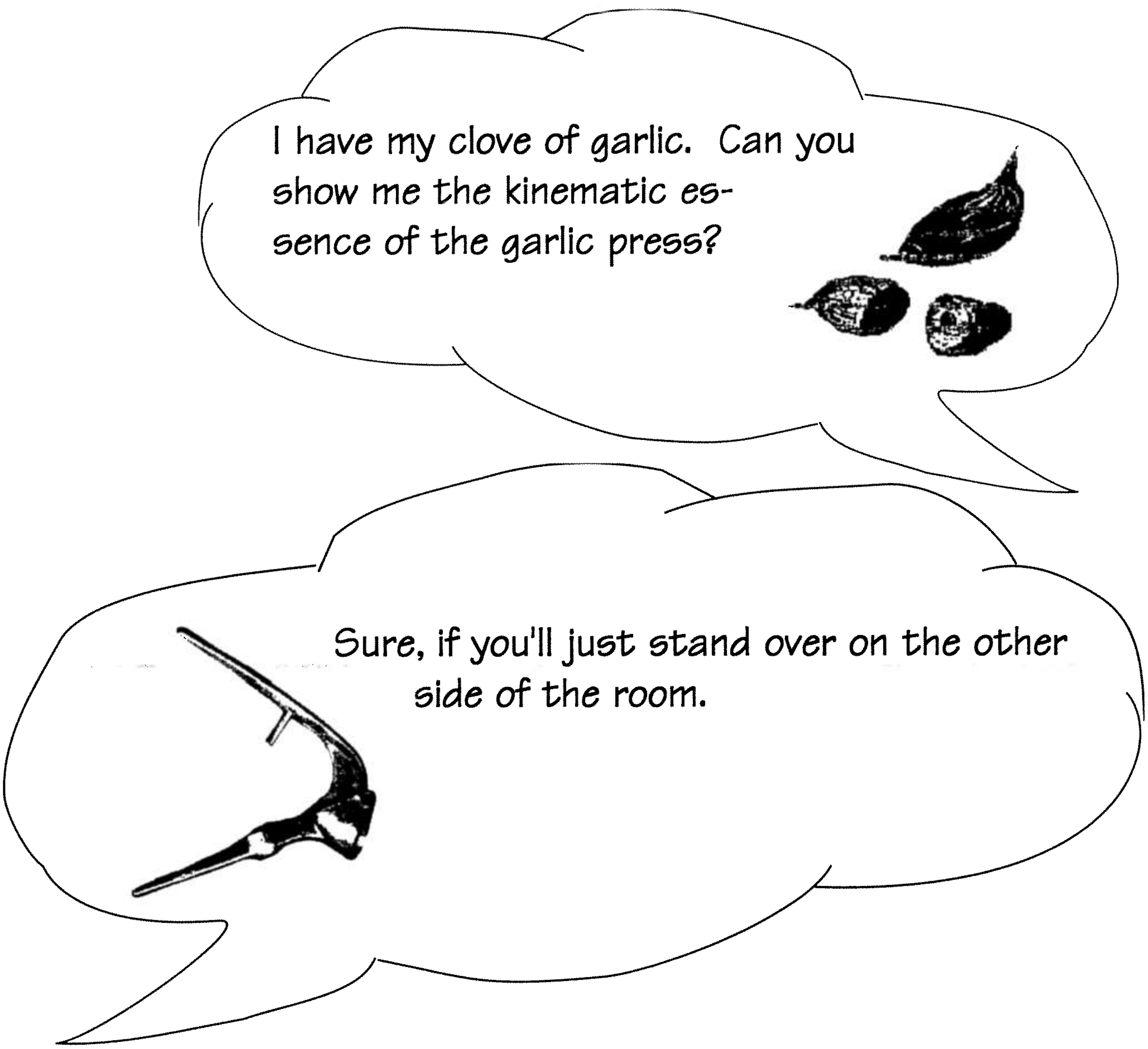
To help in describing the motion of the moving body, you can select some convenient moving reference coordinate system within the body such as that shown by the x_j y_j system shown (as seen when the body is in the jth displaced position).

This limitation to "planar motion" means that the path of any point attached to the body will always trace out a curve that lies in a plane parallel to the XY reference plane. It means that life is a lot easier for a mechanism designer. The body may actually be a three-dimensional object, but for the purpose of figuring out such things as where pivots or sliders might be attached, you can consider the object conceptually as if it were run over by a steam roller and flattened out into the XY plane. This mental mathematical exercise will help you to get many more practical mechanism designs than you would otherwise be able to find. Trust me.



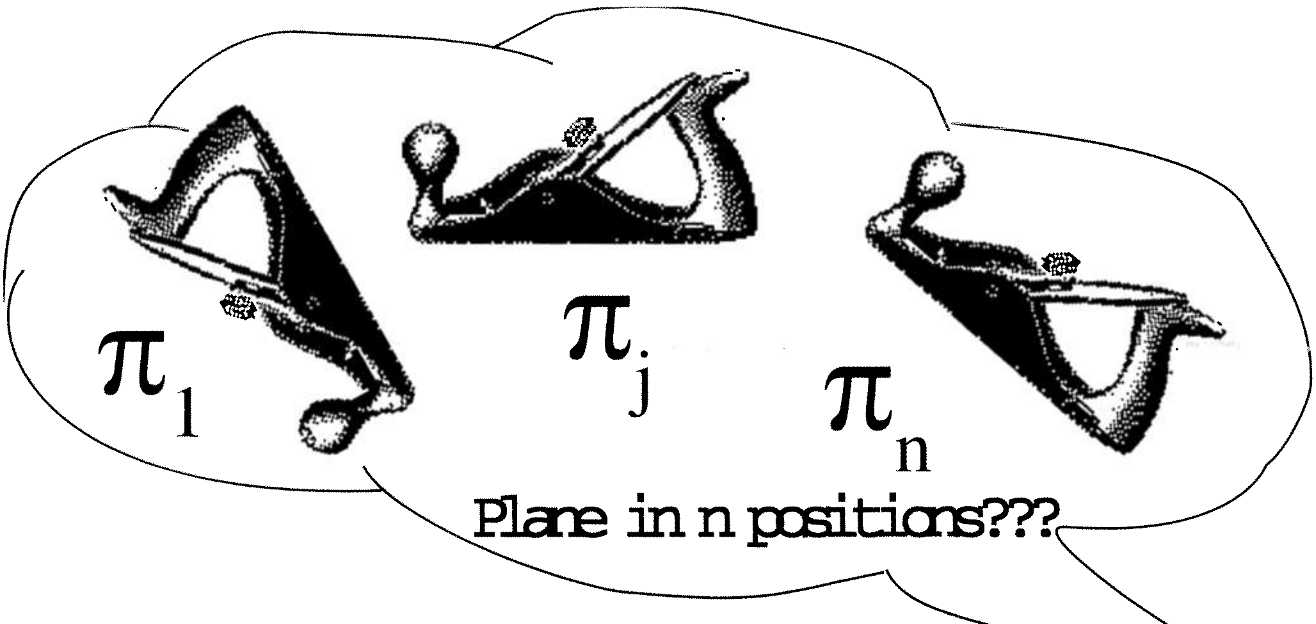
By the way—while you are considering the body as being run over by a steam roller, consider it also as having been squished out to be infinite in extent. That means you can mathematically look for possible locations for mechanism joints anywhere they might be attached and not just within the original confines of the body.

Later, when you go to actually implement your mechanism design, you can attach brackets to the squashed out picture of your moving body to make these joint locations you found mathematically into real physical points attached to the body. Then, since your object had been flattened by a steam roller, you would need to puff the body back up into 3-d for the purpose of taking it off your drawing board (or out of your computer) and making it into a real mechanical device.

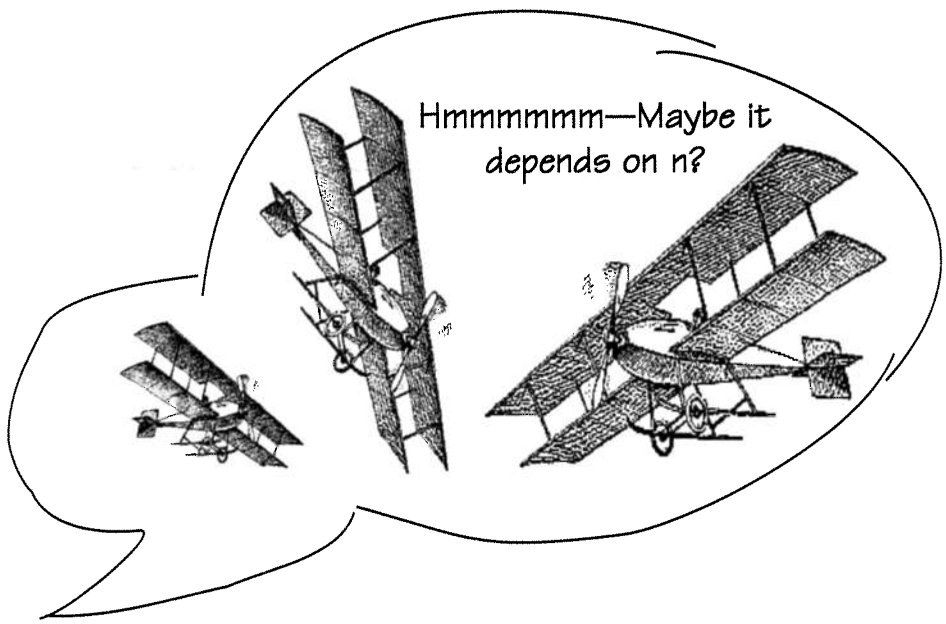


Question:

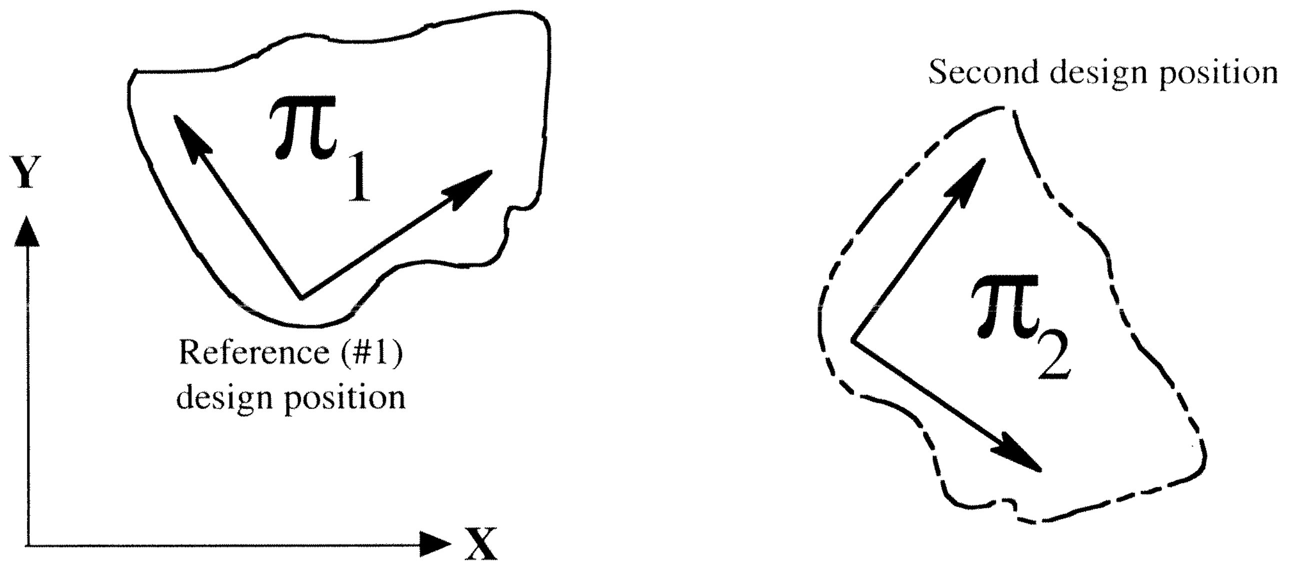
Given a plane π in n specified positions, π_1, π_2, ..., π_n, are there any points in the plane π that lie on a circle in all n positions?



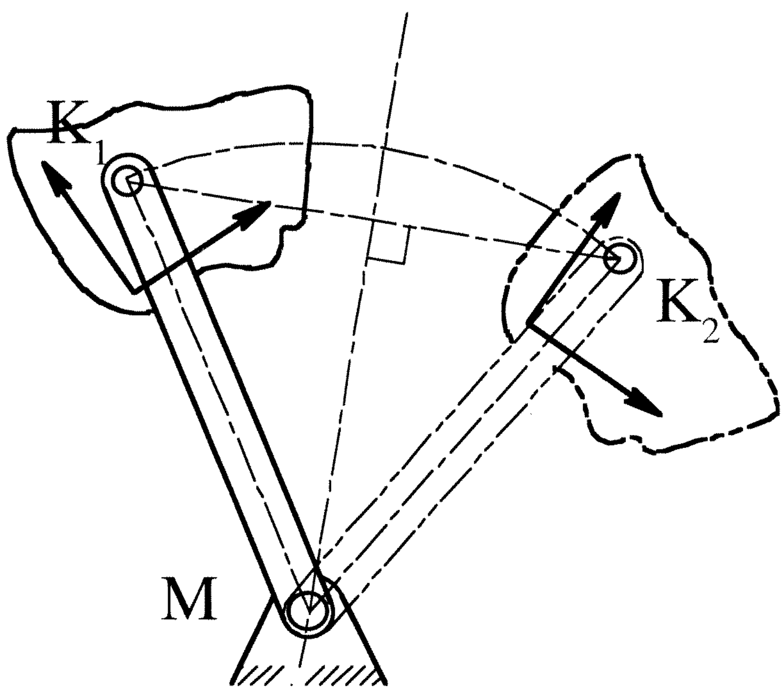
Not that kind of a plane, you idiot!



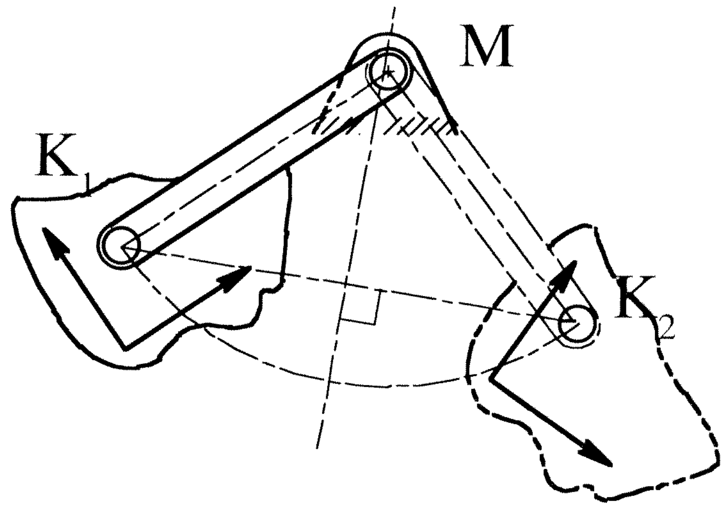
Suppose we are given any set of two design positions for a plane π. We want to design a mechanism to move the body from the first to the second position.



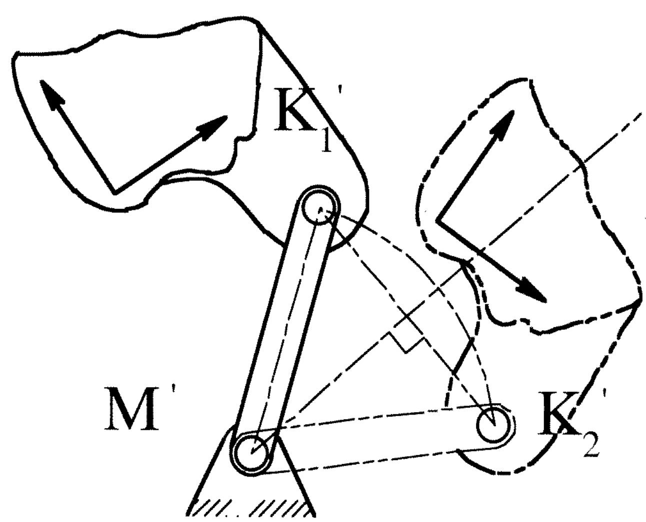
Just like the not-so-swift apprentice carpenter who nailed his shoe to the floor and asked the boss "Why do I keep walking around in circles?" links of mechanisms that are hinged to the frame like to go around in circles.



As any fool can plainly see, **<u>any</u>** point of the moving plane lies on a circle for the two arbitrarily given positions of the plane. In fact, it lies on an infinite family of circles!

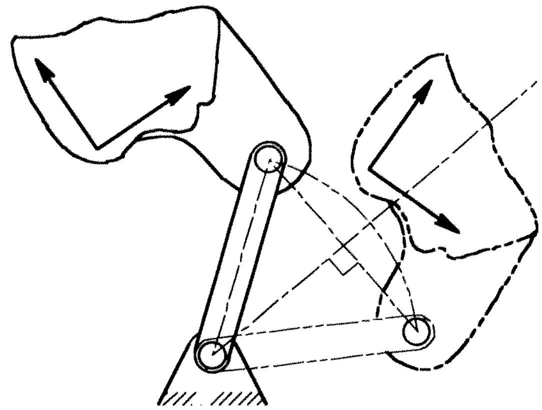


If we pick a point K in the plane and plot its location in the two given positions of the plane (K_1 and K_2) then any point M chosen on the perpendicular bisector of K_1-K_2 can be used as the center of the circle. We could guide point K from K_1 to K_2 by means of a link, pinned to the frame at M and pinned to the moving body to be guided at point K_1. Similarly, we could pick any other point of the body such as point K_1'. Plot where it goes in the second position and you can find still more possible links, as shown in the figure.

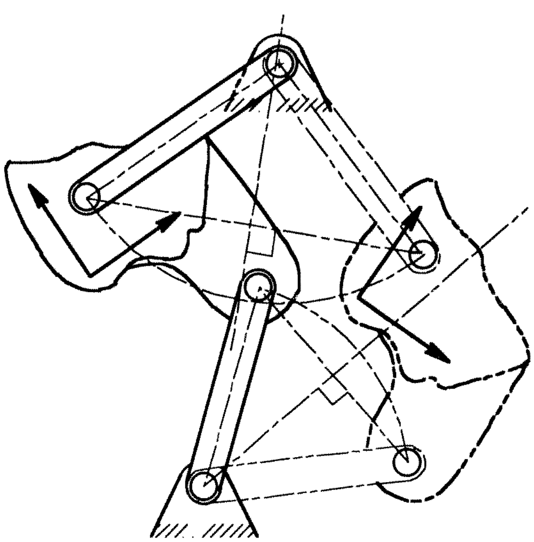


"Hold it!" (You say.) "Point K_1' isn't even a point of the object!"

"<u>**AHA!**</u>" (I say.) That's why I told you to think of the object as being squished out to be infinite in extent. You can easily attach a bracket to the object to make the point K_1' into a physical point of the object, even if it doesn't lie on the original outline of the body. So for our purposes, every point can be considered to be a possible point of every body if it will be helpful to do so. That's what I did in this figure.

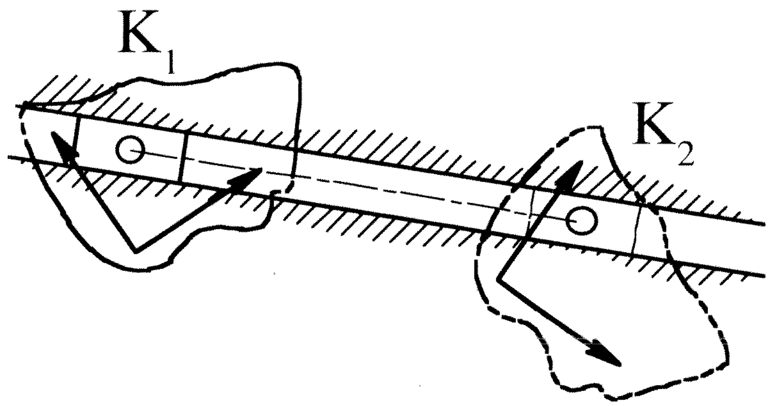


Doing the same thing for another freely chosen point of the body, we would quickly end up with a possible design for a four-bar linkage which would be capable of guiding the body **EGGZAKTLY** through the two given design positions!

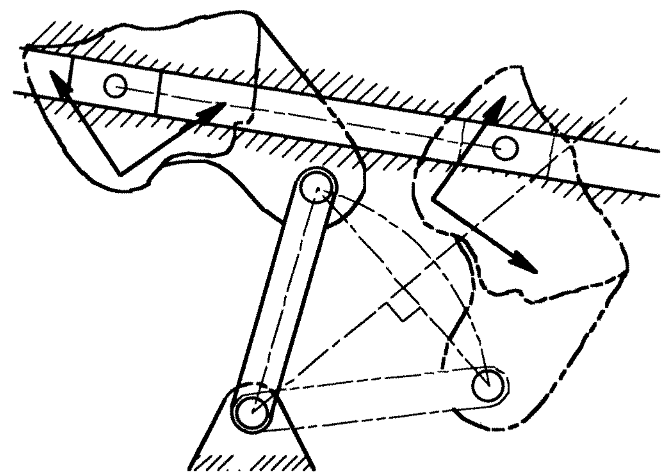


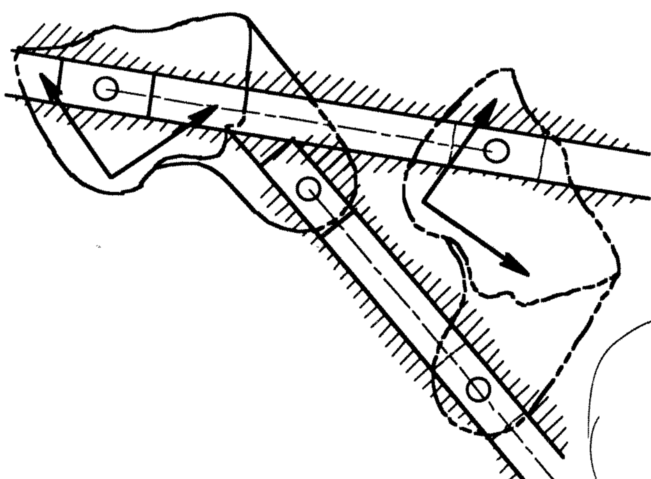
Well, strictly speaking, we aren't sure if this thing will actually <u>move</u> between the two design positions, but at least we know it can be <u>assembled</u> in both positions which is a hell of a good start to coming up with something workable. It beats standing in the shower all day trying to dream up some gadget out of the blue!

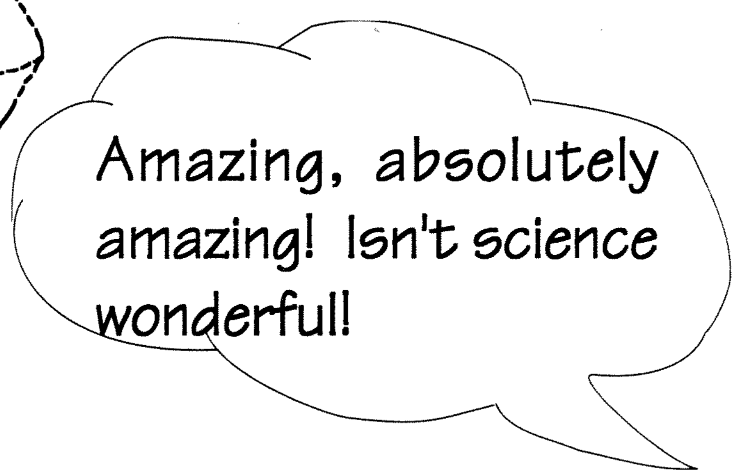
We could also guide point K from K_1 to K_2 by using a straight slider or, for those who must do things the hard way, by using an **infinitely long link.**

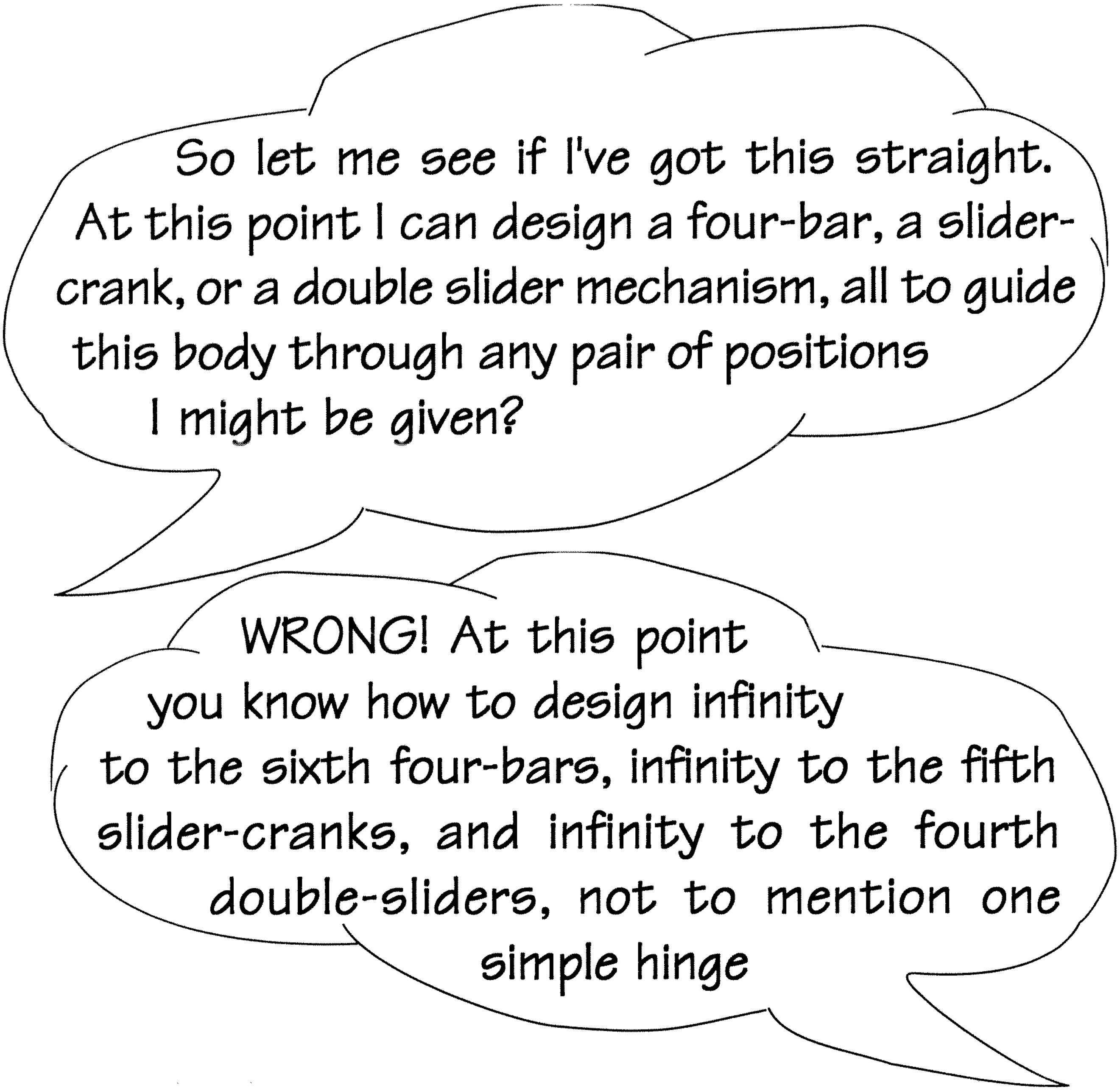


In this way, we could synthesize a wide variety of slider crank mechanisms, double slider mechanisms, and the like, all capable of exactly moving the body through the two given design positions.





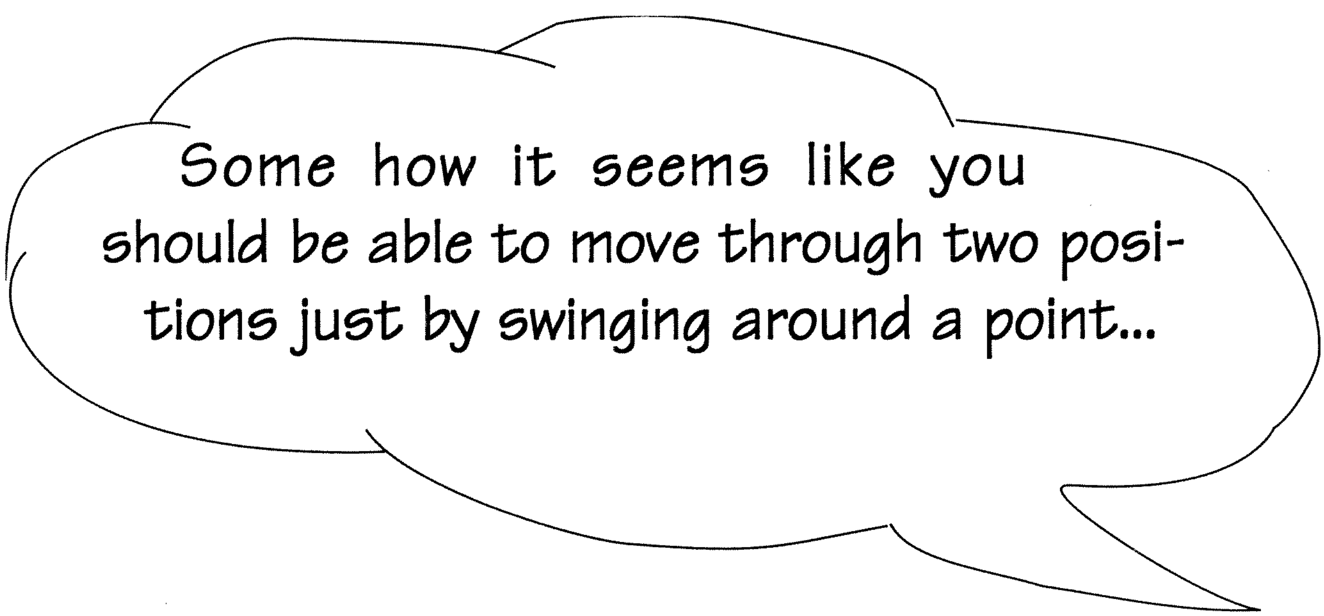


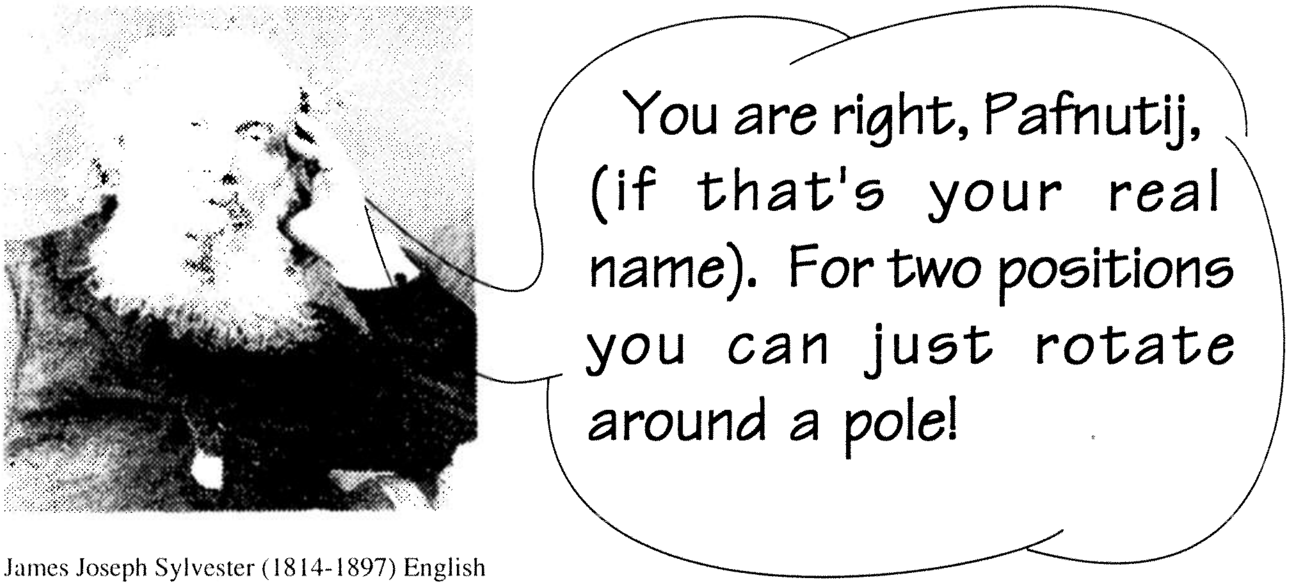


How can that be, you ask? Infinity is as many as you can have of anything, isn't it?

Not so. This is a higher-dimensional space we are dealing with. Any point in the two-dimensional plane can be used as a possible moving pivot K for a linkage. Since we can pick the point anywhere in a plane, this means we have a double infinity of choice. Picking the X-coordinate is a different infinity of choice from picking the Y-coordinate. Once we freely pick the point in one of the two design positions we can't choose where it goes in the second position- it goes wherever the given motion takes it. However, corresponding to that pair of positions, K_1 and K_2, there is a bisector line on which we can choose the fixed pivot M for one crank of our linkage. Picking a point on this bisector line gives us a third infinity of choice. If we are designing a four-bar, we have another three infinities of choice associated with the second crank link of the mechanism. Ta Dah! That gives us six infinities of choice! What that means is that a designer has a lot of design freedom which can be used to meet other design requirements and still meet the synthesis requirements of passing exactly through the two given design positions.

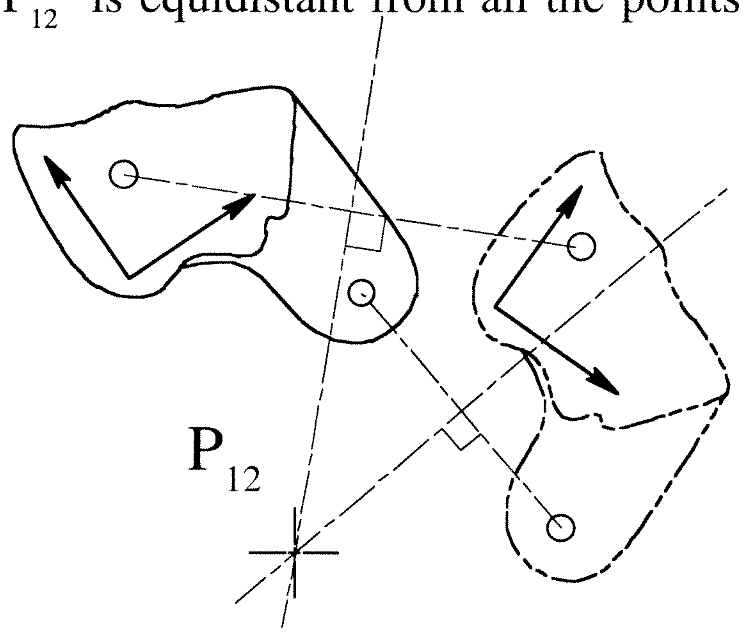
Similarly, if you are designing a slider, there is a double infinity of choice, since any point in the plane can be guided through the two positions by a straight slide track. And so forth...



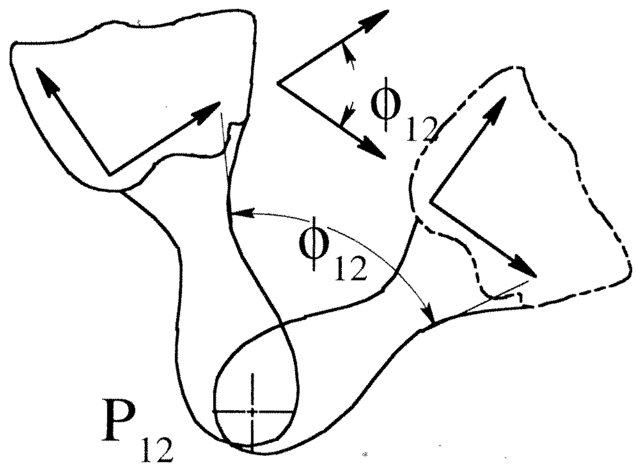


James Joseph Sylvester (1814-1897) English mathematician and lecturer on linkages.

In laying out these perpendicular bisectors to various points plotted in the two given positions, you may have noticed that they all seem to intersect at one point. The reason is that point of intersection is equidistant from all the points of the moving plane when it is in either the first or the second position. You could call the point "Arpège" but the rest of the world of kinematics calls it the "pole of rotation." Since this pole is associated with positions one and two we will label it P_{12}, although "Gucci" is a much more prestigious label. Since P_{12} is equidistant from all the points of the moving

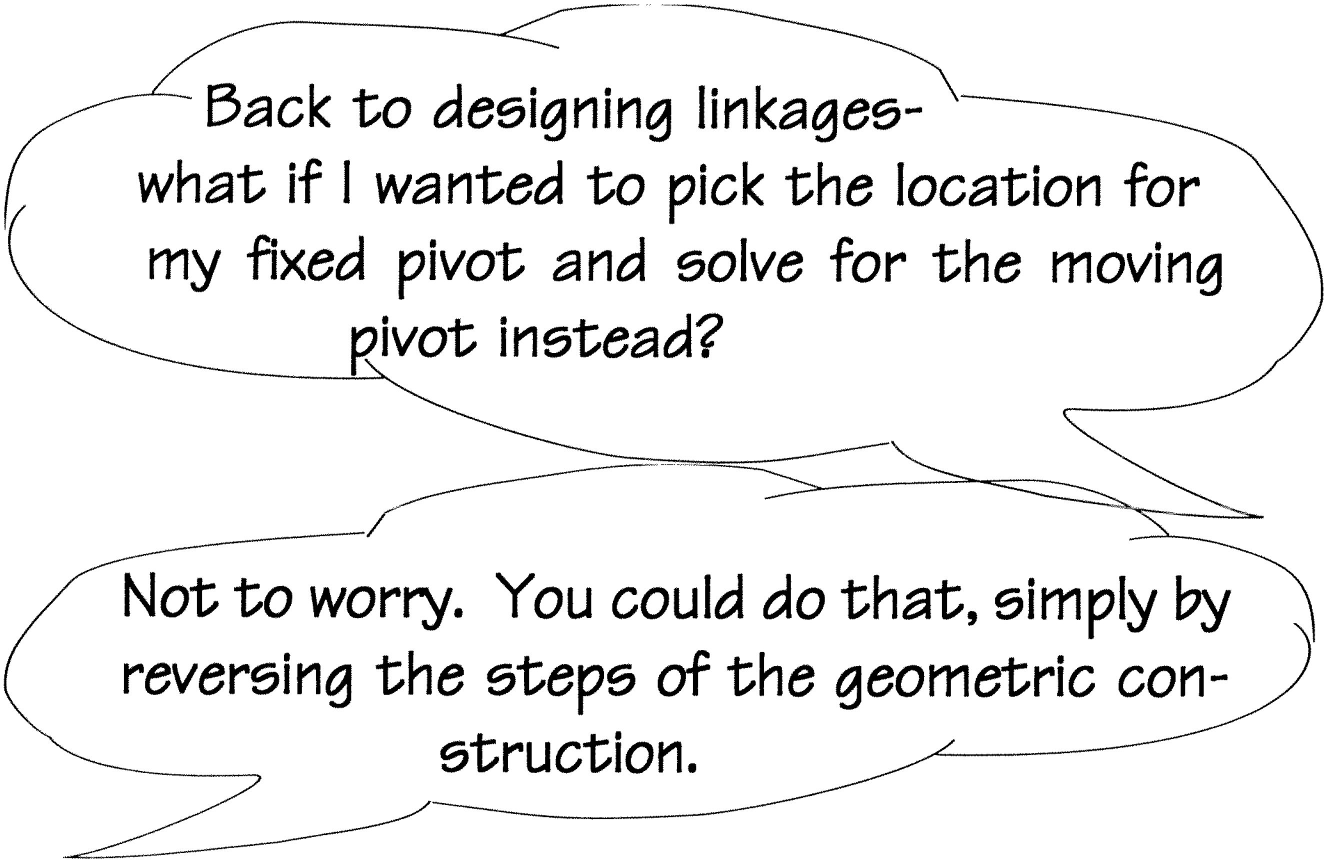


body in both positions, an easier way to guide the body through these two positions is to simply attach a bracket to the moving body extending down to the pole P_{12}. We could then swing the body about the pole by the angle ϕ_{12} (where ϕ_{12} is the angle the whole body has rotated by as it moves from the first to the second position).



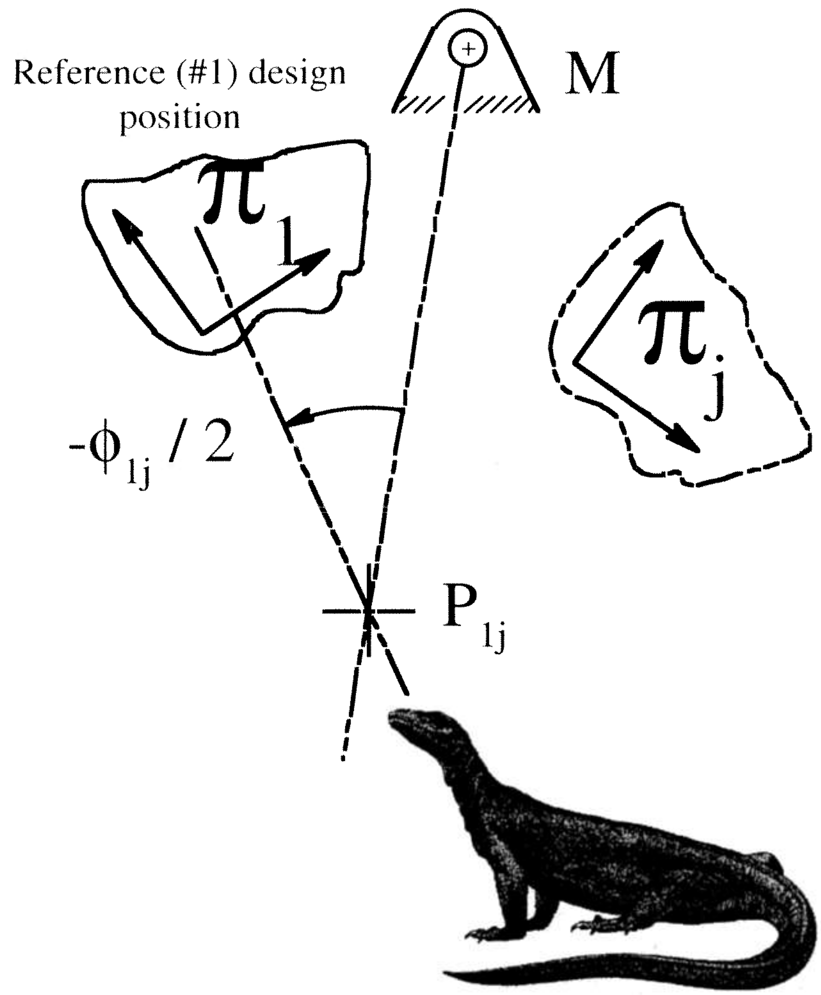
Notice that there are actually two points at the pole P_{12}. There is a point P_{12} of the moving body and there is a coincident point P_{12} of the fixed reference body. The point P_{12} of the moving body travels along with the moving body but it lies on top of the point P_{12} of the fixed body when the moving body is in either the first or the second position.

Later on you will care, since understanding these picky little details is crucial to understanding how to design mechanisms in more challenging situations!



As a moving body passes from its i_{th} to its j_{th} position, it rotates about the pole of roatation corresponding to those positions by the angle ϕ_{ij} where ϕ_{ij} has both a magnitude and a direction.

If you have a reference (#1) design position and any jth displaced position and you also have selected a tentative fixed pivot location then you know that possible moving pivots for linkages must lie in such a positon as to subtend the angle ϕ_{ij} when seen from the pole. Thus you can find a locus of possible moving pivot locations in the number one design position by the following procedure:

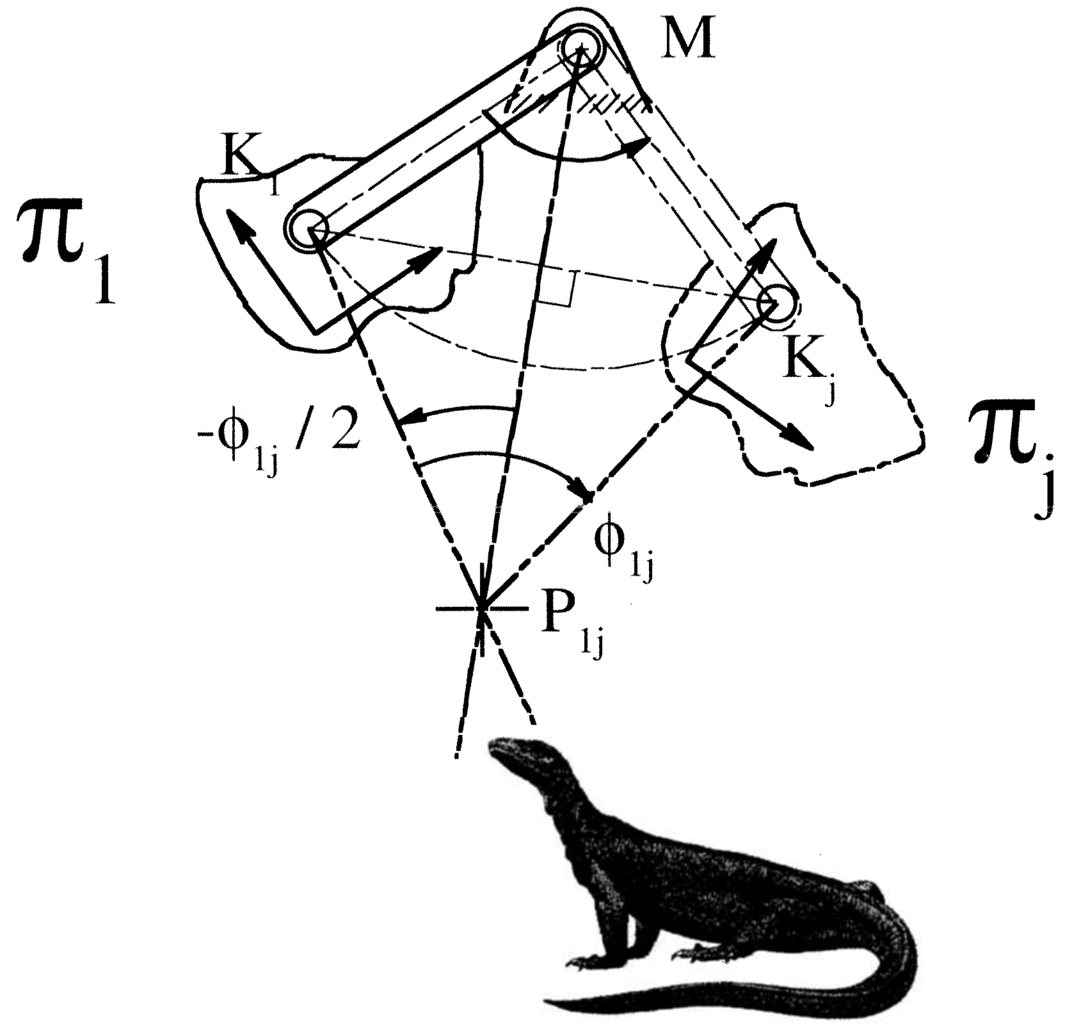


☞ Put your eyeball on the pole P_{1j}

☞ Look over at the chosen fixed pivot M

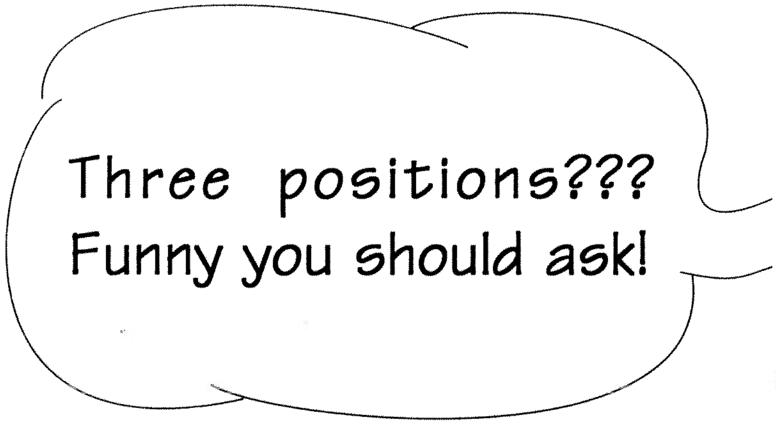
☞ Rotate your eyeball backwards by the angle $-\phi_{1j} / 2$

Anyplace on your new line of sight would be a possible moving pivot location as seen in the first design positon of the plane π.



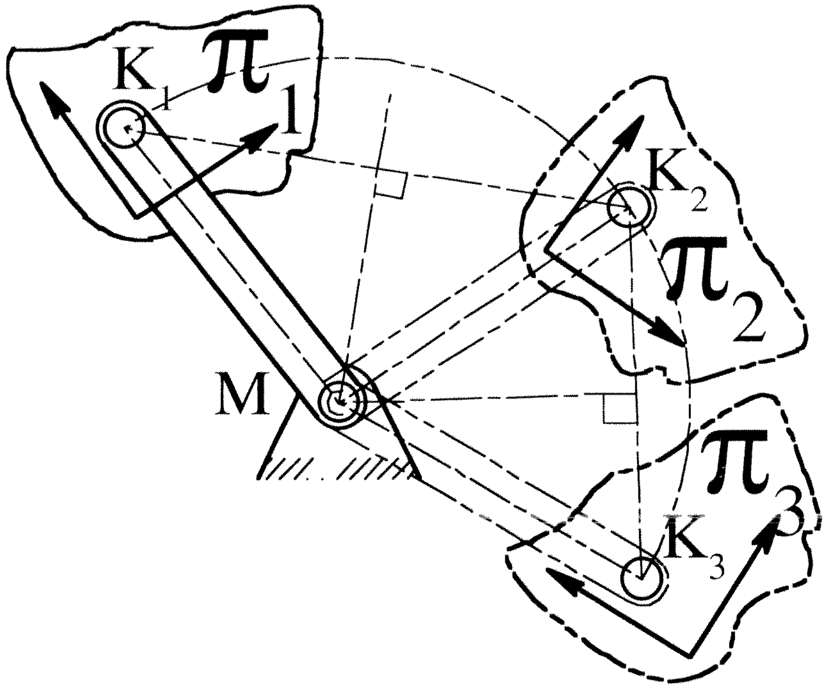
In this figure, the point K lies on a circle centered at the given point M while the body moves from its first to its j^{th} position.

Howzabout designing linkages to move through three given positions?



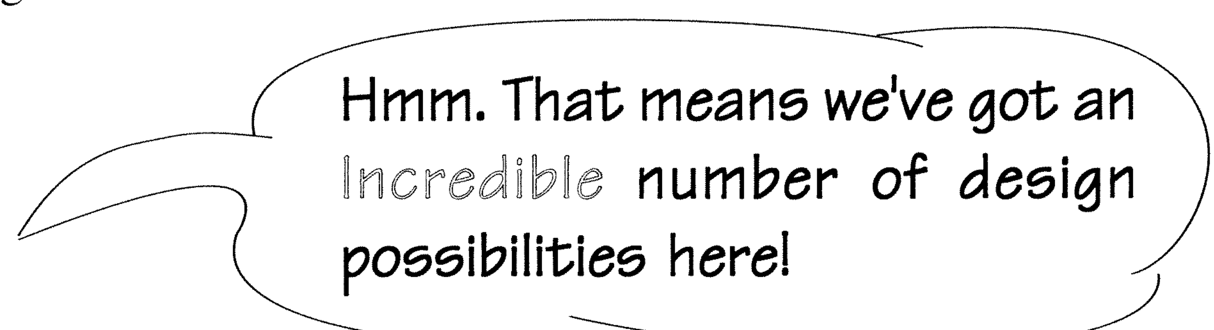
Pafnutĭĭ L'vovich Chebyshev (1821-1894) Russian mathematician who developed theories for analysis and synthesis of straight-line mechanisms. His mommy called him "Pafnutĭĭ," but his last name is sometimes transliterated as Chebyshev, Tschebytschew, Tchebycheff, Chebyshew, or in a zillion other spellings.

Three positions of a point determine a unique circle, so we can pick any point of the moving body, plot its location in all three positions, draw perpendicular bisectors to its three positions, and find the fixed pivot for a link capable of guiding the point through the three positions. Do this for any other point of the body and you have synthesized a four-bar which can be assembled in all three positions. Maybe, if you are lucky, it can even move through the three positions without locking up on the way! (To know if it will lock up you need to check to see if it satisfies some other vital conditions. For instance, you need to check its Grashof Type, and you need to see if it was properly lubricated with WD 40 or Mobil 1.)



In this figure, the point M is the center of the circle through which the chosen point K passes as it goes through the three given design displacements of the plane, π_1, π_2, and π_3. For that random choice of a moving pivot point K there is one and only one possible centerpoint M that could be used as a fixed pivot.

However, he says, there was nothing particularly magical about the choice of point K. Any point in the plane π could have been chosen as a moving pivot and a corresponding link could have been designed.



What makes design of linkage mechanisms difficult is that most of us have a hard time figuring out where what we are looking at when we stare at a stationary piece of machinery, much less something that is wiggling all over the place. In synthesizing linkages you are trying to figure out the dimensions of four-bars or slider mechanisms which will cause them to move around, gyrate, **jive**, spin, and twirl in a precisely predicted manner as they move through their paces and do their thing. This can be mind-boggling. Designing mechanisms can be addicting, but we need to learn to design these things without ingesting dangerous and illegal mind-altering substances.

To make the linkage visualization process simpler, we mentally take snapshots of the various positions through which we want the mechanism to carry the moving body. We then superimpose these photos taken at various times in the mechanism's cycle so that we can stare at a kind of "story board" showing the overall picture as to what must happen. This gives us a static display we can study. We can think about it while we stand in the shower. We can let the problem soak into our minds without being distracted by its temporal nature.

Thus, when we graphically lay out the design of our mechanism, we are doing so with the mechanism sitting at rest rather than powered up and running at full tilt. We need to lay it out as it would be seen in a snapshot taken in a fixed reference position. For consistency, we label the various positions through which the mechanism needs to move so that the reference position will always be the number one design position.

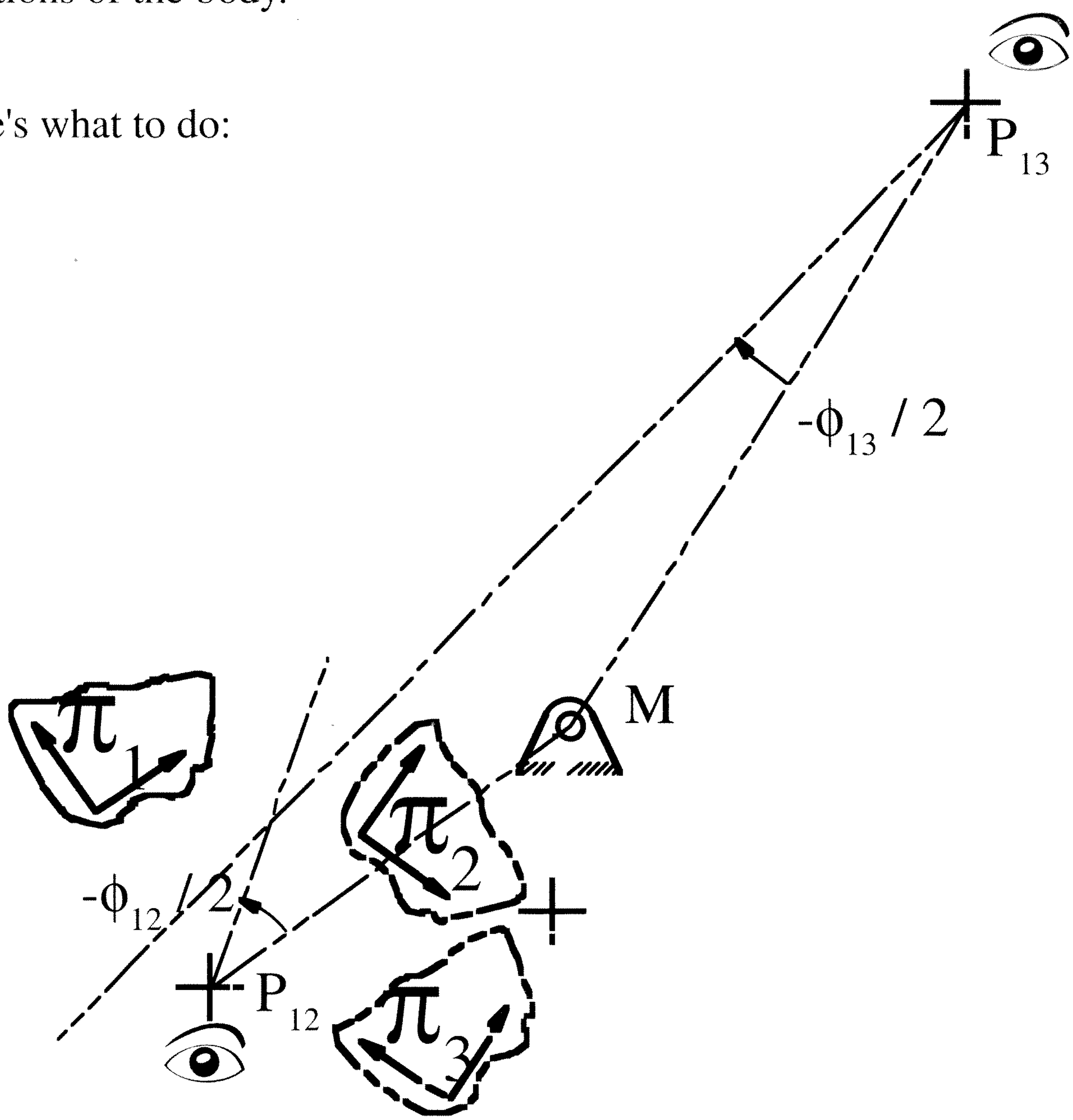
Yooo hooo! We were talking about three positions, remember? Is it possible to pick the location for the fixed pivot and solve for the moving pivot?

You can also pull a stunt like the one you did for two positions, picking the location of the fixed pivot M and solving for the location of the corresponding moving pivot K. As I just told you, however, you need to keep in mind the fact that the moving body is really moving around through the various design positions. When you carry out geometric constructions, be sure they are based on data that is valid in the #1 reference position and are not transitory phenomenae which only pertain to some other position like the second or third position.

Here's what we are trying to do: We want to find a link which will swing on a circular arc centered at the given fixed pivot as the moving body moves through three given positions. We need to find a point in the body that can do that little task.

As the body goes from its first to its second position it will rotate about pole P_{12} by the given plane displacement angle ϕ_{12}. As it goes from its first to its third position it will rotate about pole P_{13} by the given plane displacement angle ϕ_{13}. We are looking for a pivot point K which is within the body and is compatible with both of these requirements. As the body goes from position one to two to three, it will drag point K along with it from K_1 to K_2 to K_3. Further, these three points through which K moves need to all lie on a circle centered at M so that a fixed length link can connect the pivots K and M. point must lie on a circle centered on the fixed pivot M for all three positions of the body.

Here's what to do:



☞ Put your eyeball on the pole P_{12}

☞ Look over at the chosen fixed pivot M

☞ Rotate your eyeball backwards by the angle $-\phi_{12} / 2$

☞ Put your <u>other</u> eyeball on the pole P_{13} (but be careful to keep your <u>first</u> eyeball on P_{12}, since you need to maintain that line of sight!)

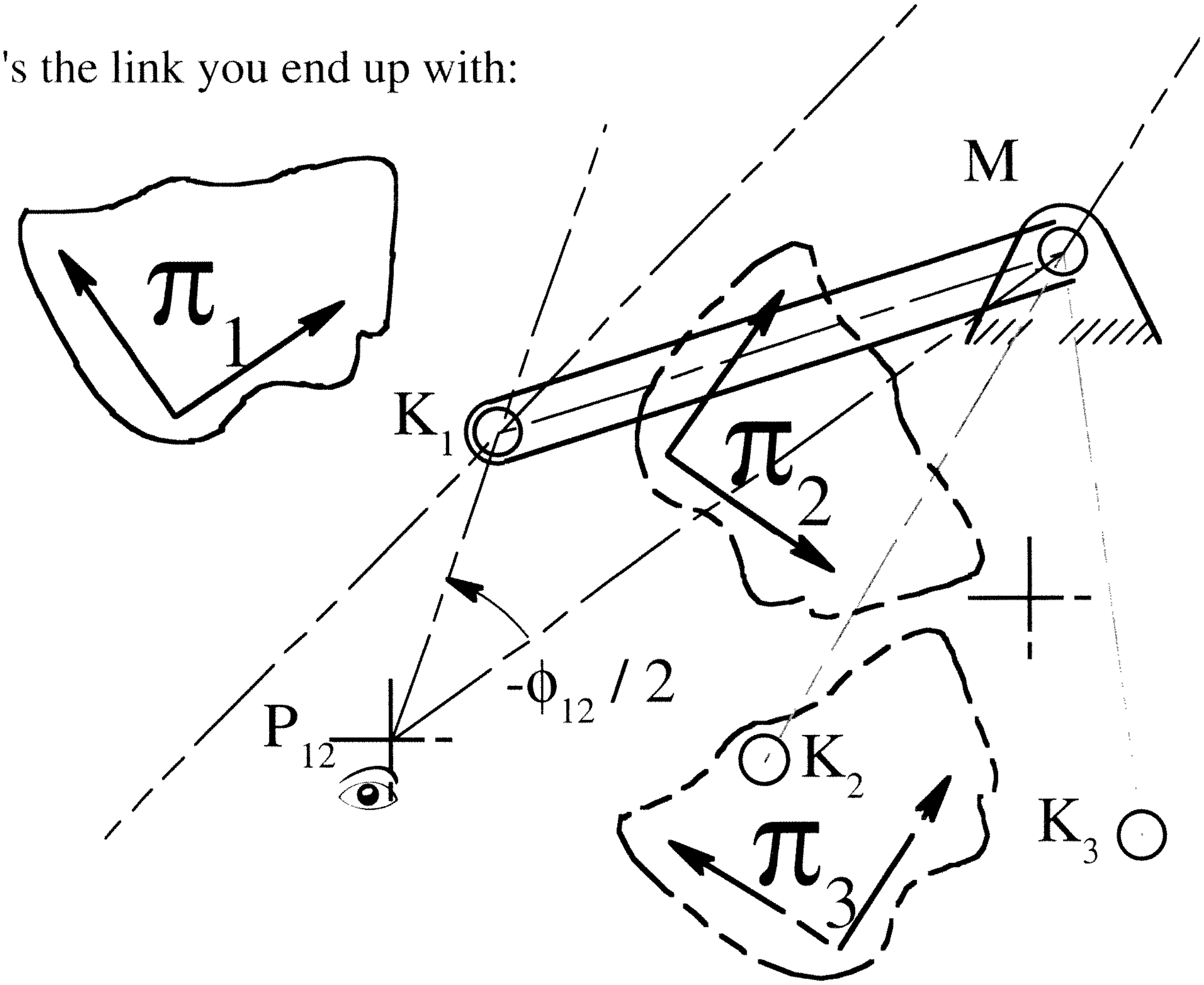
☞ Look over at the chosen fixed pivot M

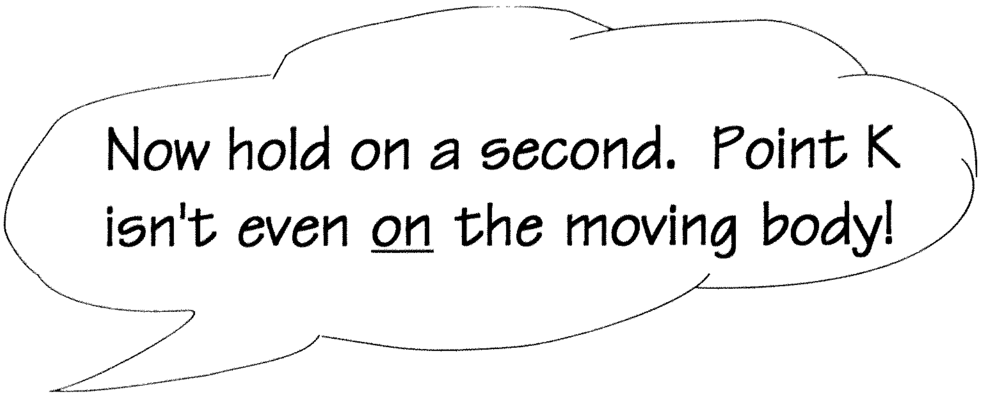
☞ Rotate your eyeball backwards by the angle $-\phi_{13} / 2$

Where the lines of sight of your two eyeballs intersect is the location of the only possible moving pivot location for that choice of a fixed pivot. This point is seen in the first position of the object, since both poles had a number one subscript and since we rotated backwards to get into the starting position. Hence and forsooth, we found point K of the moving body in its number one position. Incidentally, don't stare at the point too long, since you will get a terrible headache, having your eyeballs crossed that way.*

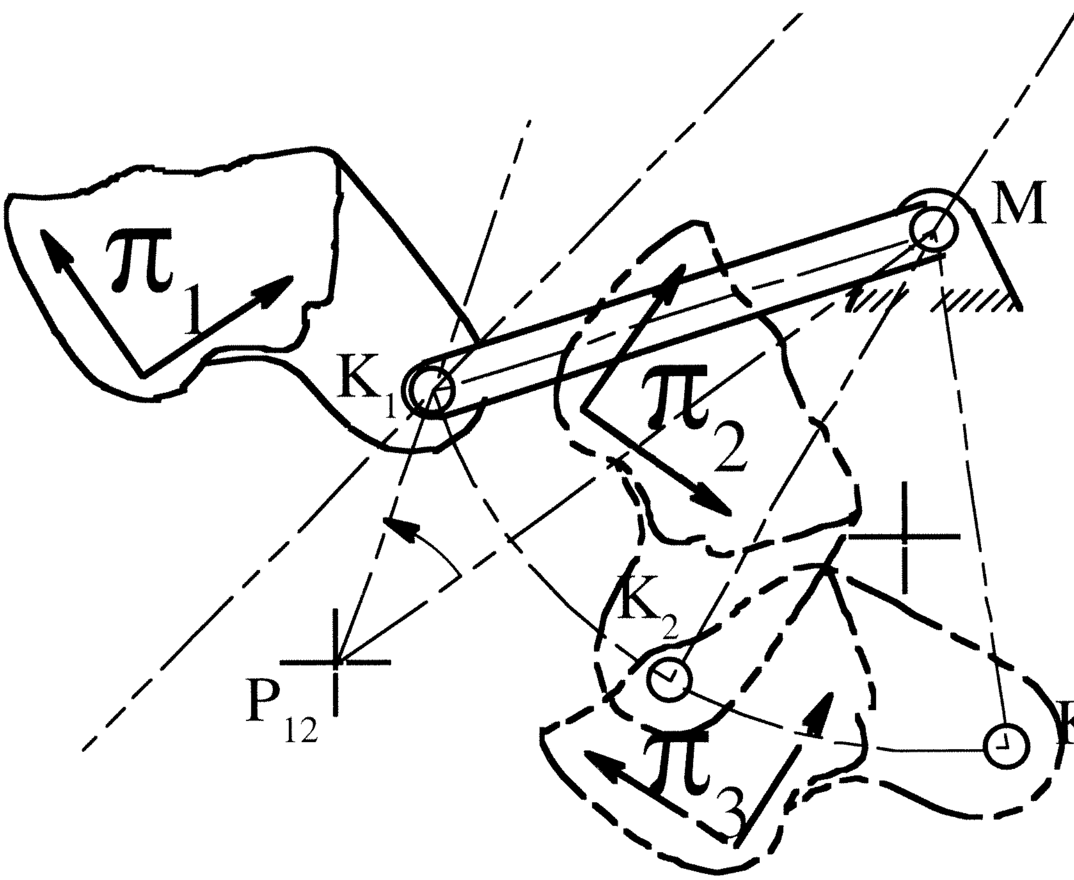
* Required health warning by the Surgeon General

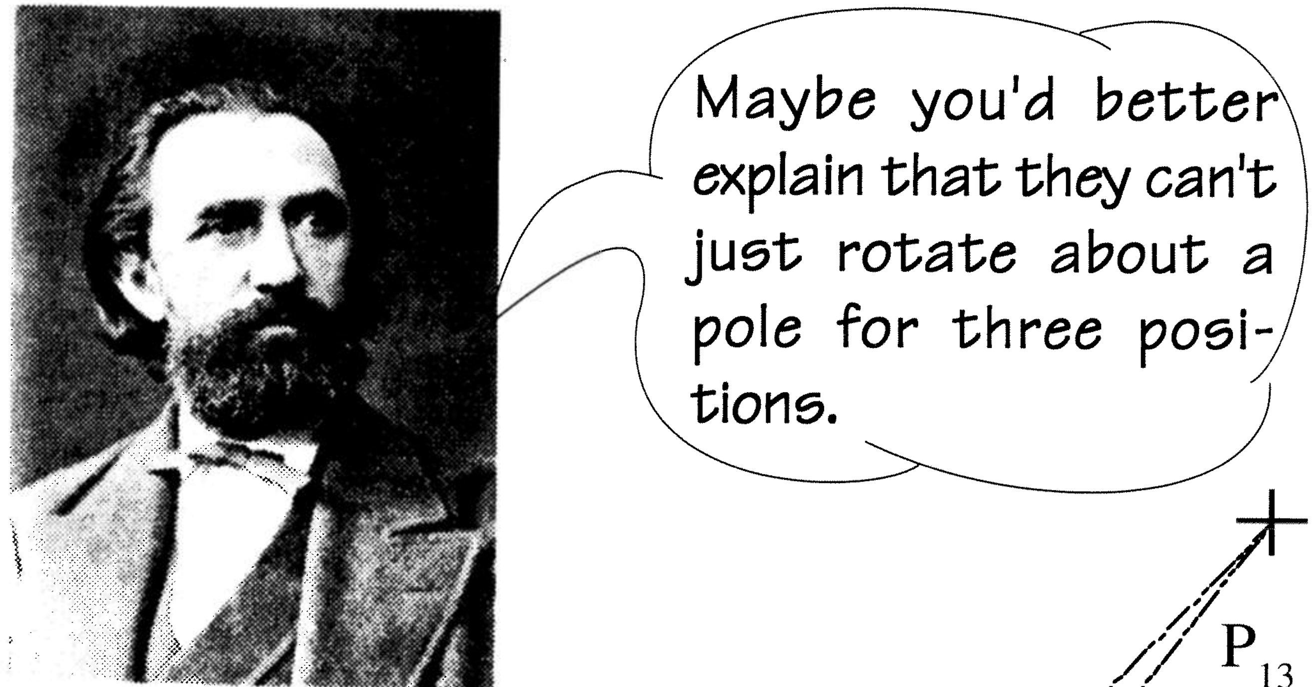
Here's the link you end up with:



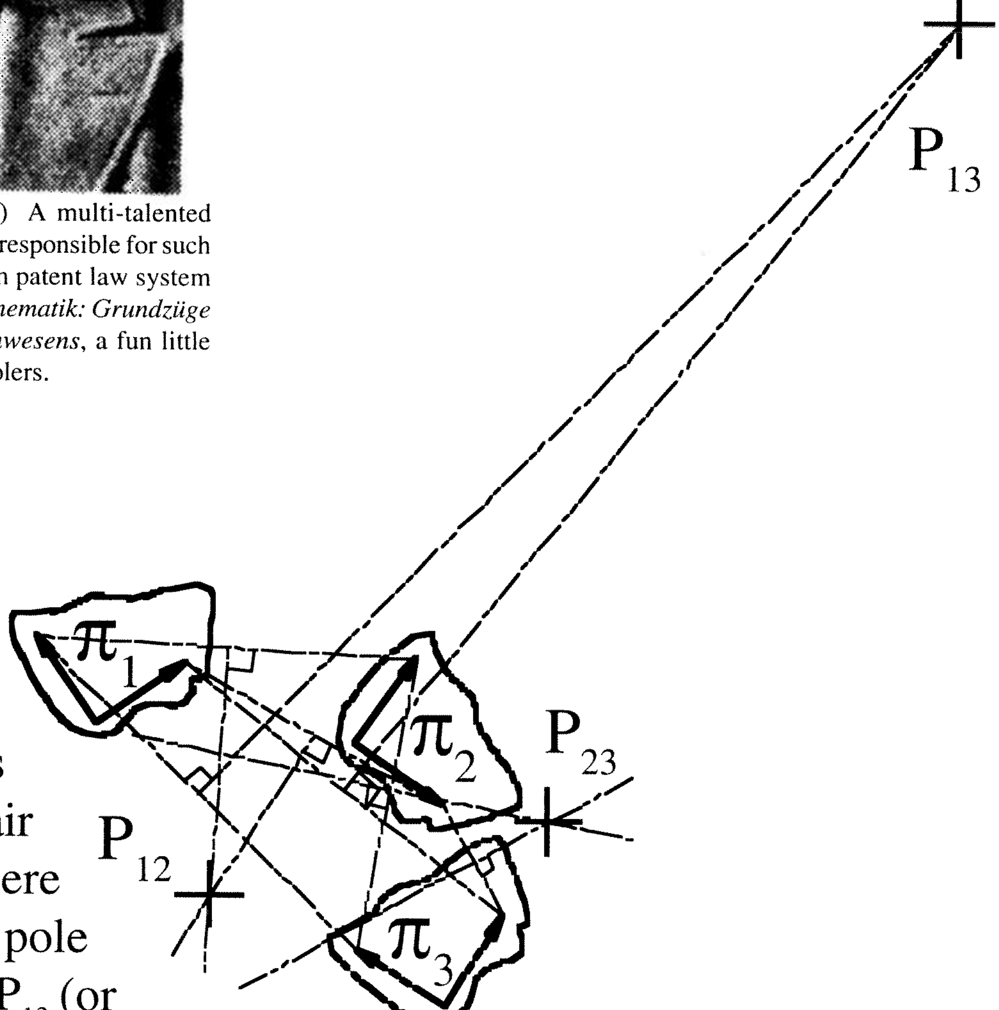


Of course it is, knucklehead! You just need to attach a bracket to the body to implement it as a point of the physical object. Mathematically, it has a constant relationship to the moving coordinate system, so it really can be considered as a point of the moving body. Remember what I said about considering every body as if it were run over with a steam roller so as to be infinite in extent?





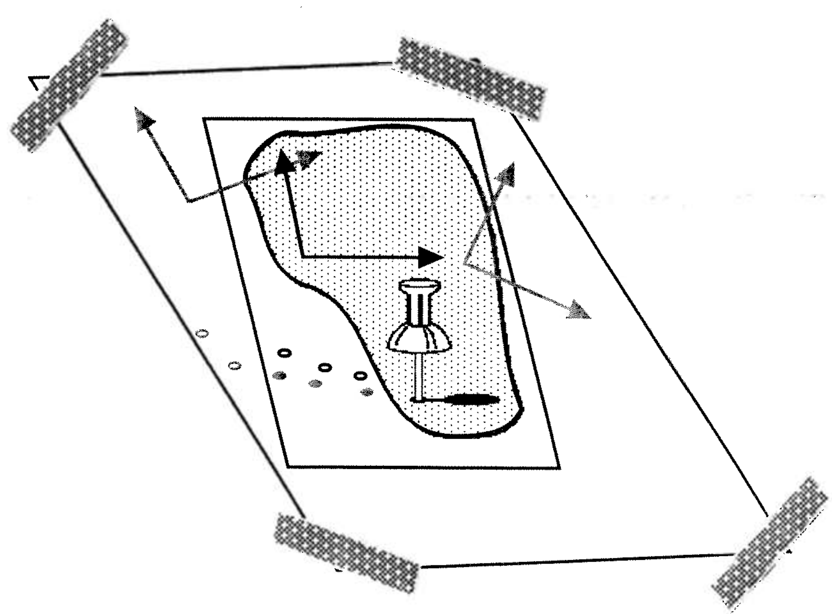
Franz Reuleaux (1829-1905) A multi-talented mechanical engineering dude responsible for such tidbits as the German uniform patent law system and author of *Theoretische Kinematik: Grundzüge einer Theorie des Maschinenwesens*, a fun little book for precocious preschoolers.



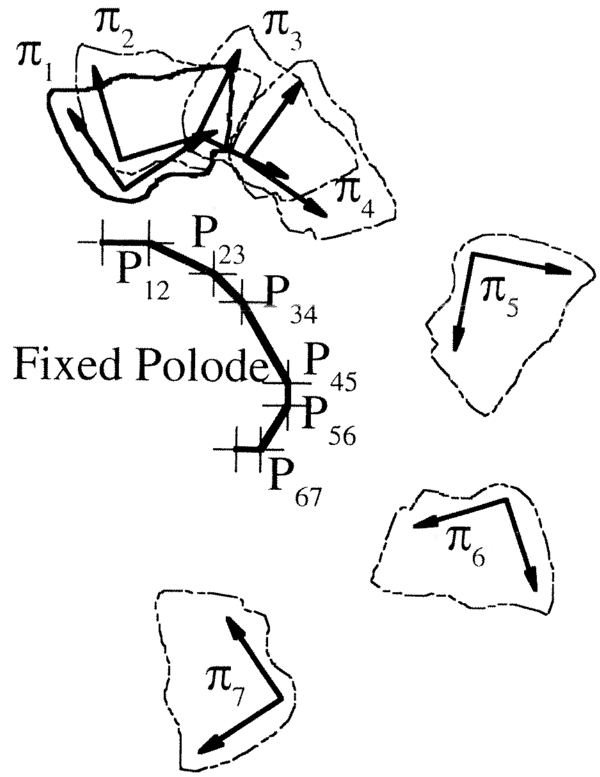
For three positions of the moving body there isn't just one pole to consider, but there is a pole for each pair of positions. There is a pole P_{12}, a pole P_{23}, and a pole P_{13} (or P_{31} if you prefer. I don't much care which subscript you put first as long as you pay me royalties for the privilege of calling it a "pole"..)

When you had just two design positions, you could simply hinge the moving body to the frame at the pole to move it back and forth between the two positions. In these days of high labor costs, no one can afford to hire someone to run around pulling the hinge pin out of the point P_{12} and putting it in at P_{23} just to move the body on from position two to three, and then to race over and move the hinge pin to P_{31} when it is time to move the body from position three back to position one again. This is just prohibitively costly in a high speed repetitive operation, so most companies use a linkage to achieve the described effect but without having to pay union wages and benefits.

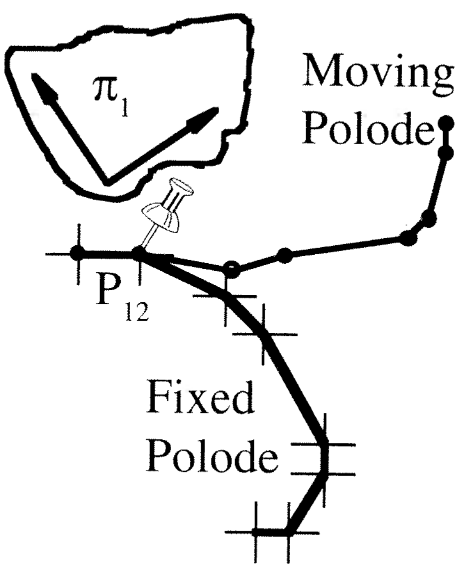
By the way, while we are on the subject of poles, notice once again that there are really two points at each pole. There is a point on the frame at the pole and there is another point on the moving body at the location of the pole. You can think of the moving body as if it were a tracing paper overlay moving around on top of a fixed sheet of paper. When we swing the body about a pole, we are effectively hinging the point of the moving body to the corresponding point of the fixed body. The point P_{ij} of the moving body is coincident with the point P_{ij} of the frame when the body is in ei-



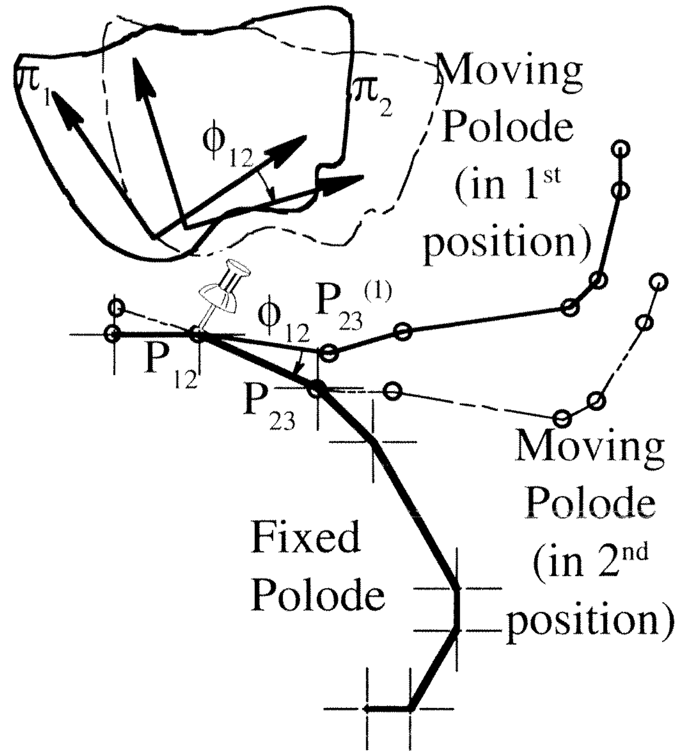
ther position i or j, but it moves along with the body and has an "image" somewhere else when the body goes to another position, just as the holes left by the push pin move around with the overlay.



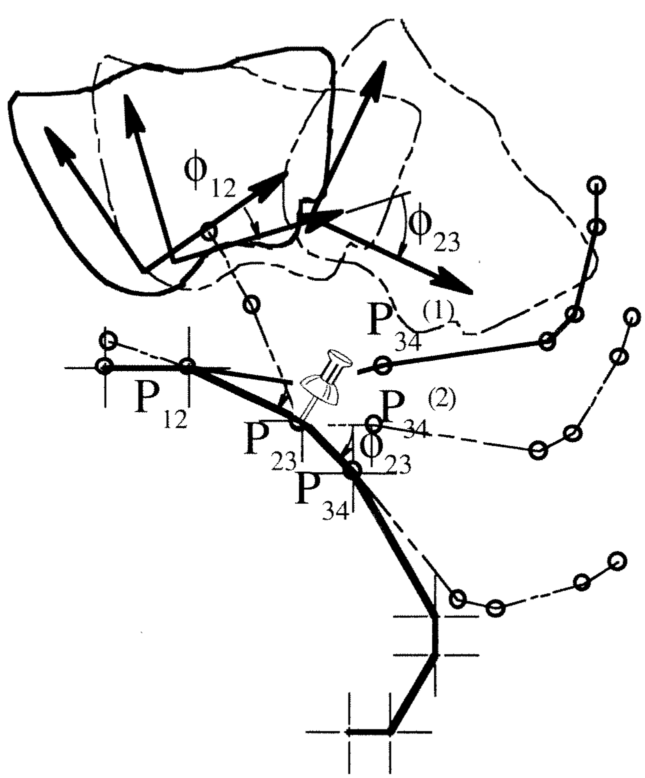
Consider what happens as a moving body passes through some arbitrary number of given positions π_1, π_2, and so forth. The body rotates about each of the poles P_{12}, P_{23}, P_{34}, ... in sequence as it does so. Imagine we are using a push pin and a tracing paper overlay on a drawing board to study the motion. If you "connect the dots" left on the fixed body by the push pin you will end up with a line called the "fixed polode" drawn on the reference plane. Similarly, if you "connect the holes" left on the moving body by the push pin you will end up with a line called the "moving polode". This line is drawn on the moving plane and travels around with the body. Here is what it would look like as seen in the number one position of the body for the case of the seven positions picked above. There is a "pinhole" on the moving polode corresponding to every one on the fixed polode. In the number one posi



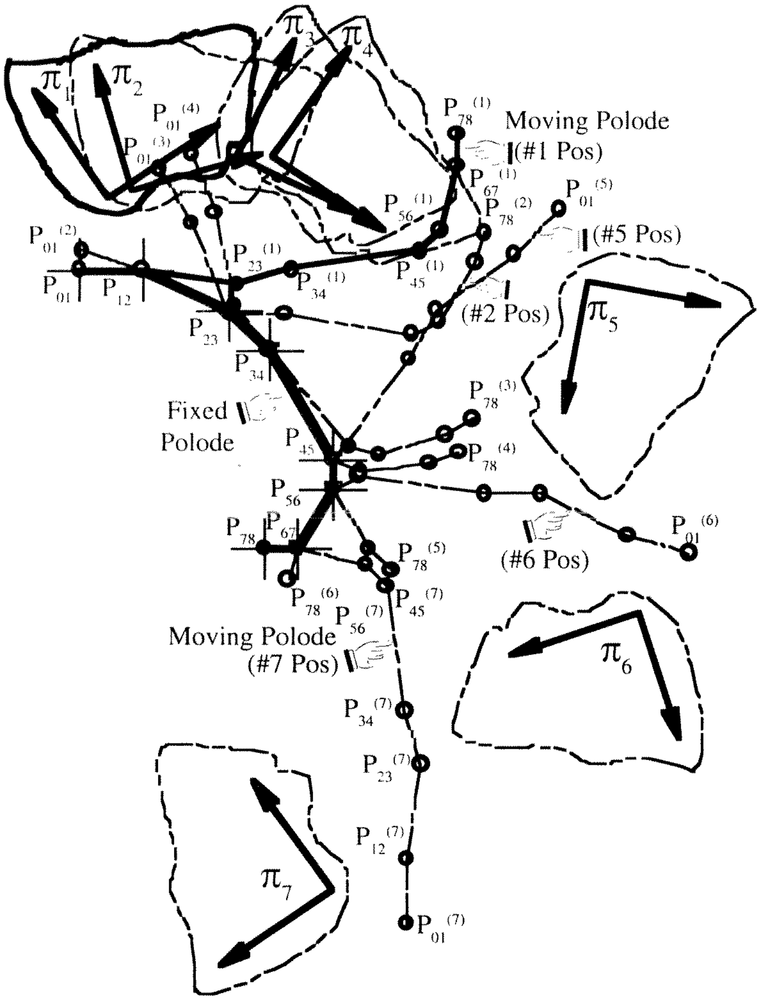
tion shown, the pinhole corresponding to the moving pole P_{12} is lined up with the pinhole for the fixed pole P_{12}.



As the body swings from its first to its second position, it rotates about P_{12} by the angle ϕ_{12}, the given displacement angle of the moving object. The moving polode moves along with it and the next pinhole on the moving polode falls into position over P_{23}, the next pinhole on the fixed polode. Thus, the point on the moving polode that moves into position to be the pole for positions two and three is $P_{23}^{(1)}$, pole P_{23}'s *image* in position one.



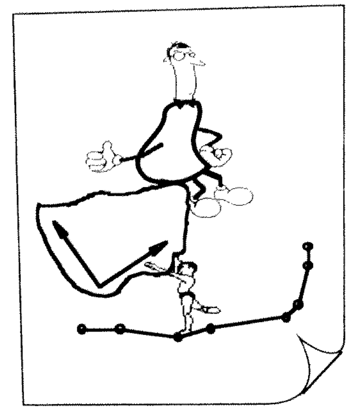
As the body moves from position two to position three, it rotates about the pole P_{23} by the angle ϕ_{23}. In this process, the point of the moving body which will be P_{34} moves into position from its image location in the second position, $P_{34}^{(2)}$. When the moving polode was back in its first position this point had an image, $P_{34}^{(1)}$.

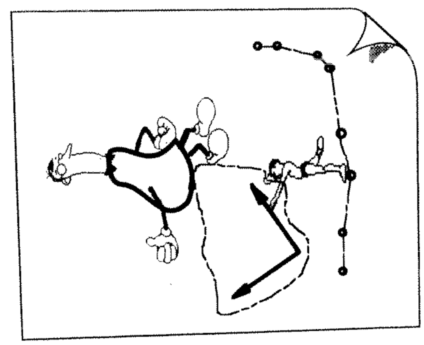


For the seven positions of the moving body, this is what the polodes look like. All the points of the fixed polode have images in each of the other positions of the moving body. (A 0th position pole has been added at the start and an 8th position pole at the end, to illustrate how the tails of the curves might look at the start and the end of the given motion.)

Notice that the moving polode rolls without slipping on the fixed polode as the body moves through the given sequence of positions. If you wanted, you could guide the body through all the positions by simply rolling (or flopping!) the moving polode on the fixed polode. You can use this principle to build a practical device in many situations, but you need to take into account possible interferences due to such things as acute angles and folds in the polodes. To avoid politically incorrect interferences you should make sure that every time there is a cute angle it is balanced by a homely angle.

The moving polode seems to move about to an observer on the fixed ref erence frame, but it remains in a stationary position from the viewpoint of the two yokels in the picture who are joy riding on the moving body. In fact, from their point of view, the *moving* polode is *fixed* and the *fixed* polode is *moving*. Picturing how things might look from the viewpoint of an observer riding on one of the moving links of a mechanism is a very useful visualization technique in designing linkages and is called **kinematic inversion**. Fixed bodies appear to move and moving bodies appear to be stationary under a kinematic inversion, but the relative motions between parts remain unchanged. It's real *Alice in Wonderland* material.

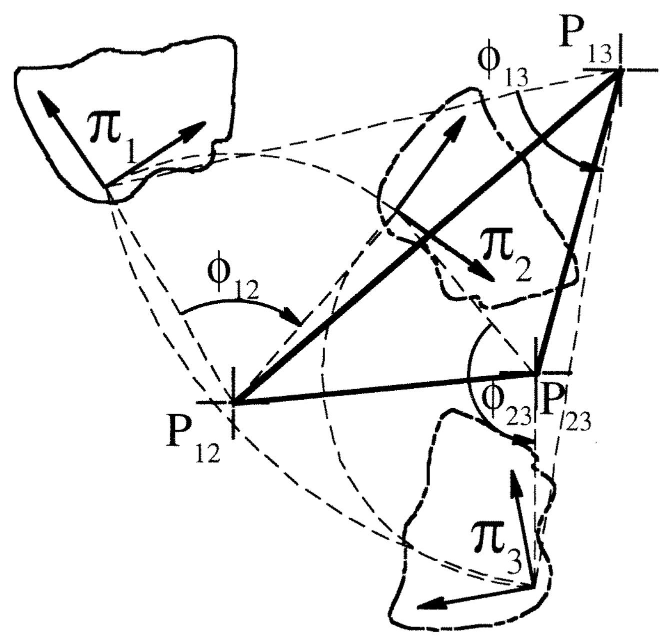




All these poles for seven positions are giving me a headache. I just wondered about something simple like three positions!

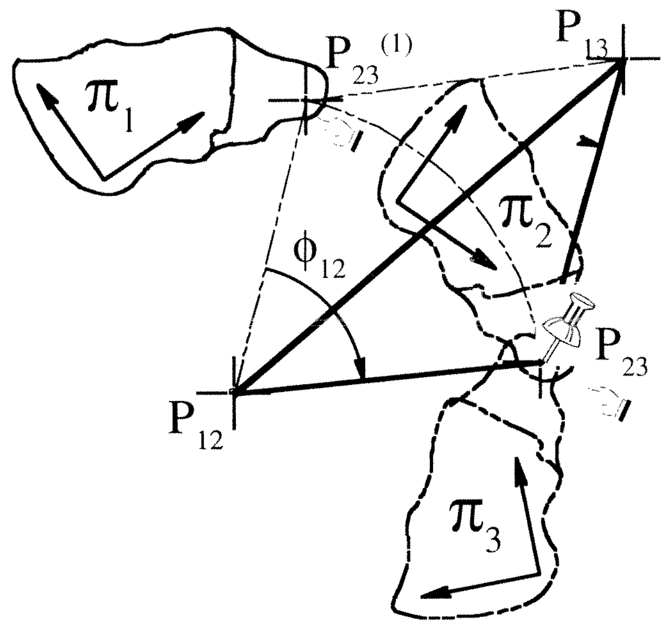
You think three positions are trivial? Ho boy are you in for a rude awakening!

Three positions of a plane define a curious little three-cornered figure called a "Pole Triangle." This little beastie is called a pole triangle probably because it has three vertices, each of which is a pole. But this is just idle speculation on my part. I really don't know for sure why it was named a pole triangle.



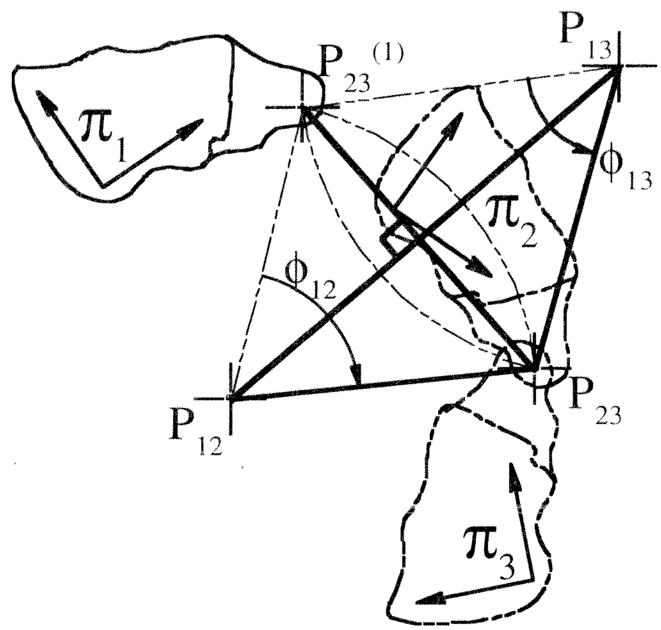
Be that as it may, the pole triangle is host to myriad weird and wonderful properties that belie its unassuming initial impression. For instance, did you know that the interior angles of the pole triangle are one half the corresponding angle of rotation, perhaps with 180° thrown in? Aha! I suspected you didn't know it!

How can that be, you ask. Aw. Go ahead and ask. The answer will help you to understand how to design slider mechanisms and also four-bars that will move through four given design positions.

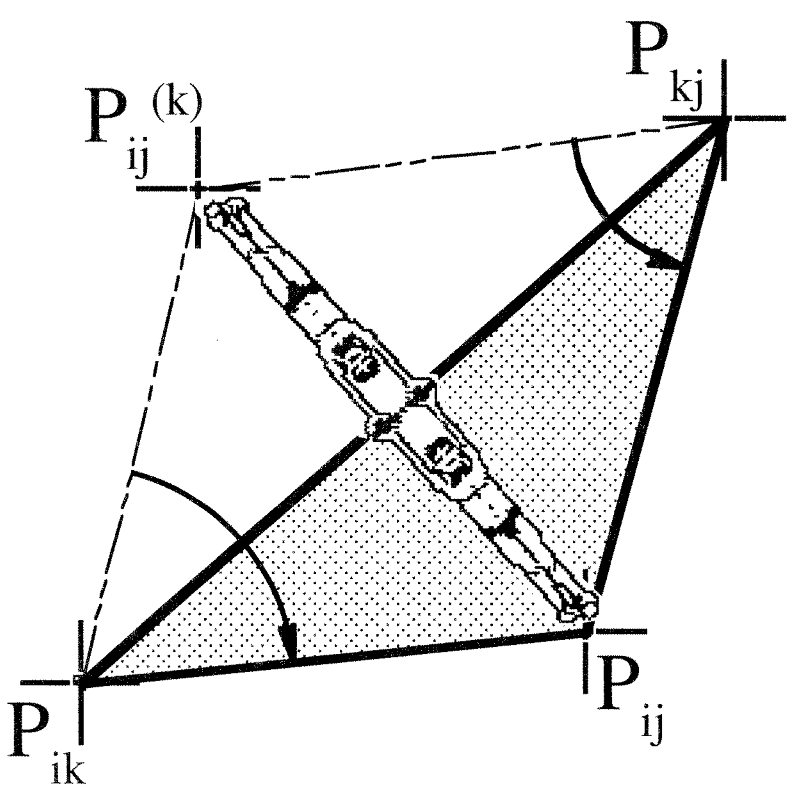


Take a careful look at the pole P_{23} as a point of the moving body. This figure shows that you could attach a bracket to the moving body, hinge point P_{23} of the moving body to the coincident point P_{23} of the fixed body and in that way guide the body from position two to position three. Fair enough. This is just a rehash of what you already know. Just to see if you have really been paying attention, let me ask "What happens to the point P_{23} on the moving body when the body is in *not* in position two or position three but is back in position one?" (Take a careful gander at the figure before you answer.) The point on the bracket that is the pole for positions two and three started out at $P_{23}^{(1)}$ when the body was back in its first position. Point $P_{23}^{(1)}$ on the bracket then rode along with the bracket as the body moved to position two. Along with all the other points of the body, it swung around the pole P_{12} by the angle ϕ_{12} in order to get from position one into position two.

Another way to think about all this is that the point P_{23} is a point of the moving body in position *three*. How did it get from its first position to position three? It rotated from its starting position, $P_{23}^{(1)}$ by swinging about P_{13} by the angle ϕ_{13}.



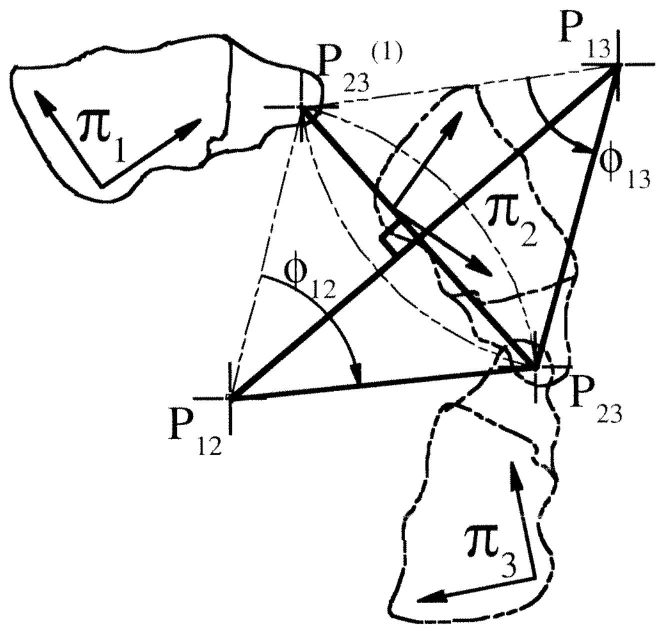
So $P_{23}^{(1)}$ can get over to P_{23} by at least two different ways. We can think of it as having rotated about P_{12} by ϕ_{12} or as having rotated about P_{13} by ϕ_{13}. As you stare at this figure, notice that the line $P_{23}^{(1)}$ — P_{23} is the perpendicular bisector to the line P_{12} — P_{13} . The point $P_{23}^{(1)}$ is really the *mirror image* of the point P_{23} taken about the line P_{12} — P_{13}.



In fact, whenever you have an i[th], j[th], and k[th] design position, you can find the *image* of the pole P_{ij} in its kth position by simply reflecting it about the line P_{ik} — P_{kj}.

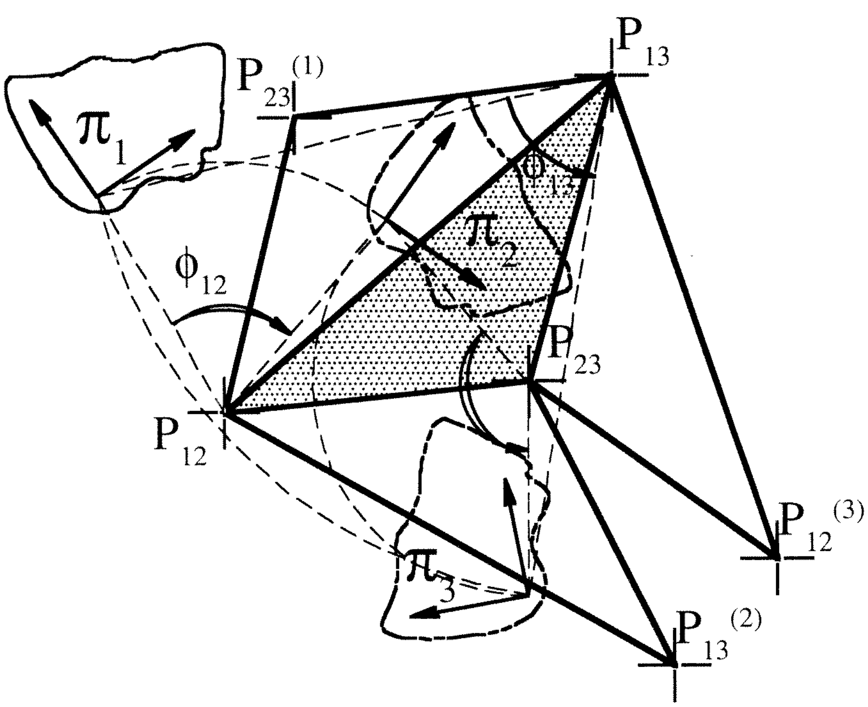
Thus the triangle whose vertices are P_{12}, P_{13}, and $P_{23}^{(1)}$ is simply the image of the pole triangle as seen in its number one position.

From this figure you can easily see that the interior angle of the pole triangle at pole P_{12} must be $\phi_{12} / 2$ and that the interior angle at P_{13} is $\phi_{13} / 2$. By similar reasoning you can convince yourself that the interior angle at P_{23} must be $\phi_{23} / 2$ since it was a random choice I made when I decided to call position one the first position instead of calling it position two or three and letting one of the others have the honor of being the



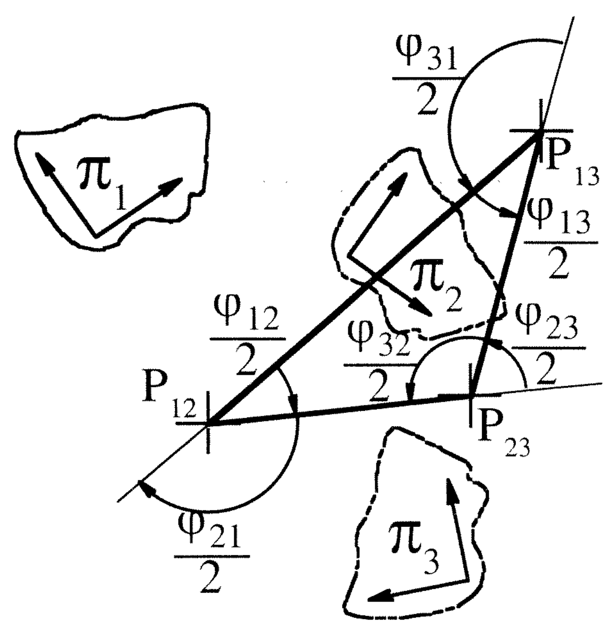
Gala #1 Design Position.

Just as pole P_{23} has an image $P_{23}^{(1)}$ in the first position (found by reflection about P_{12}—P_{13}) P_{12} has an image $P_{12}^{(3)}$ in the third position (found by reflection about P_{13} — P_{23}) and P_{13} has an image in the second position, $P_{13}^{(2)}$ (found by reflection about P_{12} — P_{23}).



There are three image pole triangles corresponding to the three design positions. In the first position we have the triangle with vertices P_{12}, P_{13}, $P_{23}^{(1)}$. In the second position we have the triangle with vertices P_{12}, P_{23}, $P_{13}^{(2)}$. Last but not least, in the third position we have the triangle with vertices P_{13},P_{23}, $P_{12}^{(3)}$. We can get these triangles simply by folding the pole triangle over along the edge which has as a common subscript the position in which we want the image drawn.

Small tidbit of trivia for you to mull on. You can get from position i to position j by rotating either clockwise or counterclockwise about the pole. (I know this is a hard concept for those of you who were raised with digital watches.) Similarly, you can get from j back to i either by continuing around in the *same* direction, making a 360° sweep, or else by going back the way you came. Half of 360° is 180, according to the old politically incorrect way of teaching math. Half of the rotation from i to j is $\phi_{ij}/2$ which, coincidentally, is the interior angle of the pole triangle at P_{ij}. This means that the exterior angles of the pole triangle are equal to $\phi_{ji}/2$ as shown on the figure below. Whether you go clockwise or counterclockwise, if you rotate from position i to j, then from j to k, and finally from k back to i you have made a 360° sweep. When I was in high school, they taught that 360° was twice the number of degrees in a garden variety triangle. Even in these inflationary times I think that is still true.



I can't imagine what earthly use any of this can have.

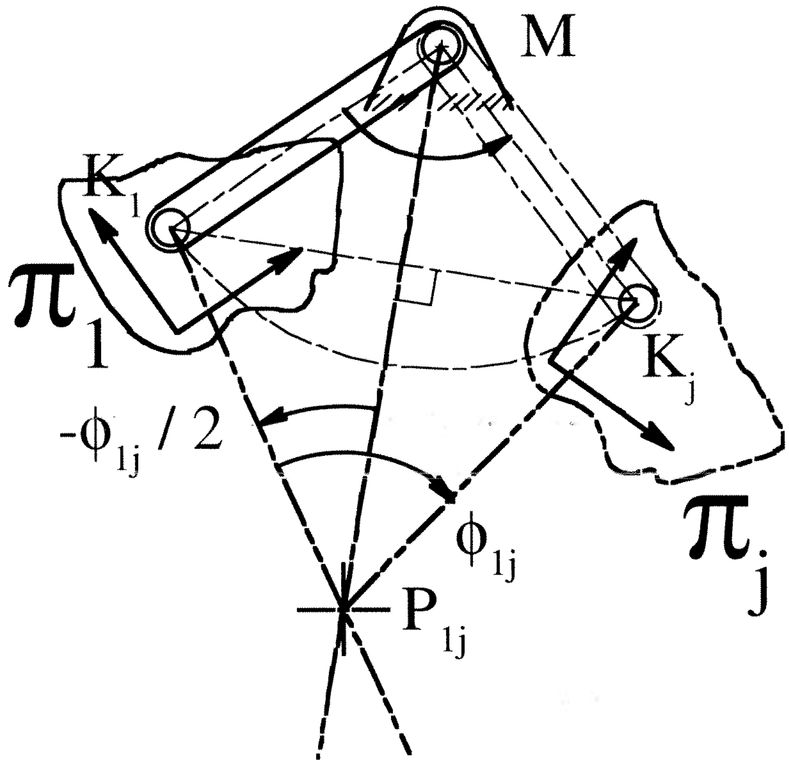
Ya wanna know what use it has?? I'll show you what use it has. It has lots of uses. I thought it was obvious to even the most illiterate reader, but if I need to spell it out to you, here goes.

Suppose you want to find a point which could be a pivot for a slider for three given design positions.

There's no way you could do that. Two positions of a point define a straight line so any fool can tell you that no slider can guide a body through three arbitrarily specified positions.

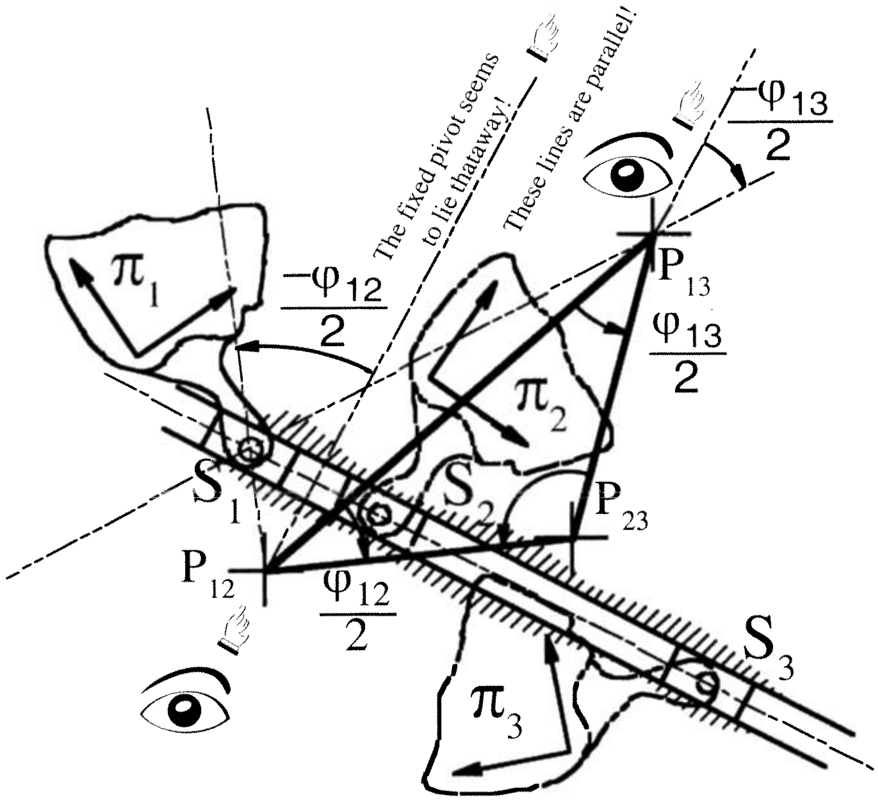
That's where you're wrong! We're not talking about just any fool here! We're talking about one who's a college graduate.

Remember how we found a possible moving pivot for a link when we knew where the fixed pivot lay? We looked towards the fixed pivot from the pole and then we looked back by the angle $-\phi_{1j} / 2$ to see where the corresponding moving pivot must lie in its starting position.



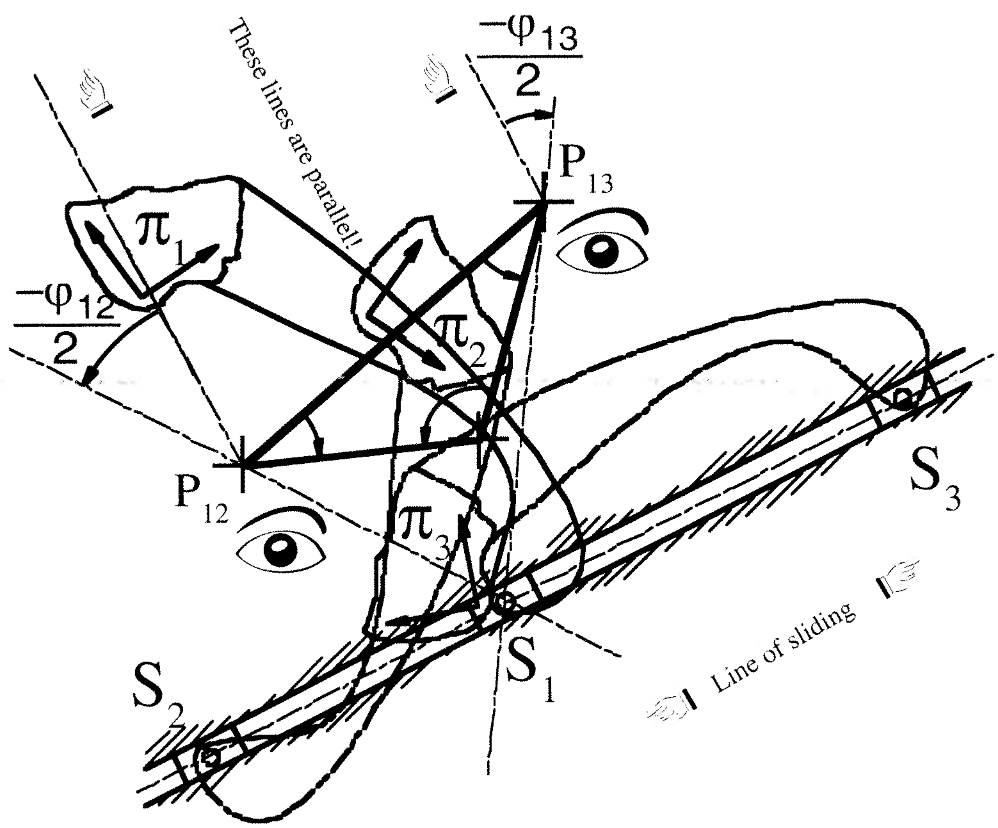
This construction would hold true even if the fixed pivot were off at infinity (which is quite a ways off, even given frequent flyer miles). If M were chosen at infinity, the point K would move on an arc of infinite radius as it moved from its first to its j^{th} position. To put it in terms a prune could understand, *it moves on a straight line.*

Since the point K is now a sliderpoint, let's give it a witty and obscure name like "S" instead of the dull, drab old "K" we were using. The "S" can be our secret code to indicate the point is a SSSSlider and to throw casual readers off the track. Most observers who skim through this section will probably assume the S stands for "Subinfeudation", the granting of part of an estate by a feudal vassal to a tenant who thus assumes all responsibilities.

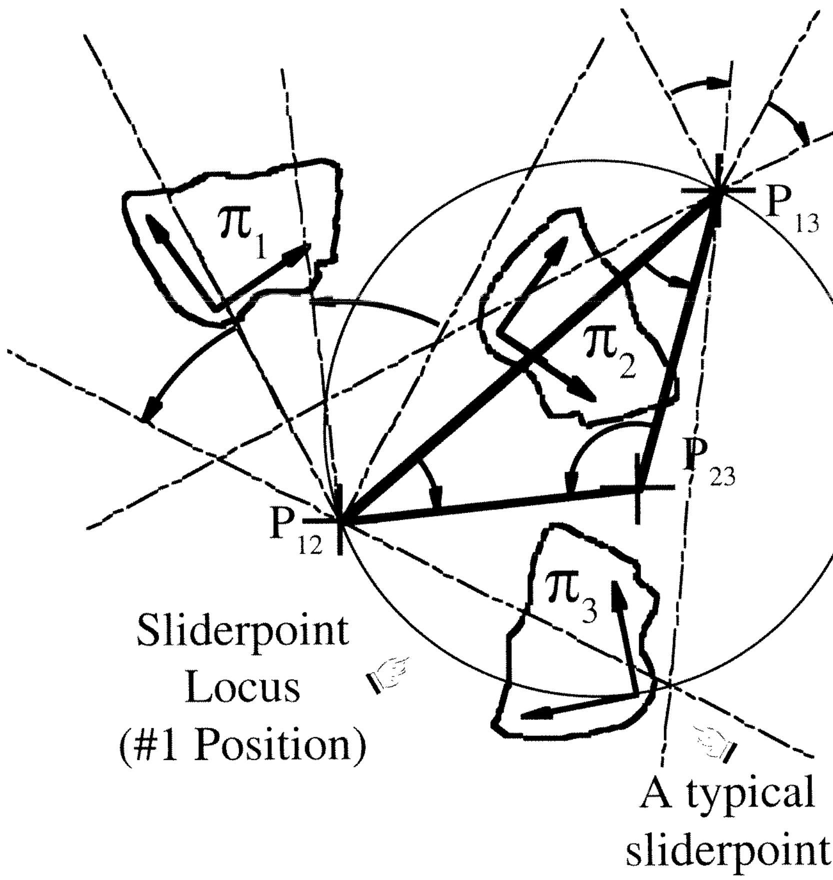


If point S is going to move on a straight line through three given positions, then the theoretical fixed pivot M must be seen at the same point at infinity regardless of whether you are looking at it from pole P_{12} or from pole P_{13}.

You can pick any direction you like and imagine the fixed pivot lying at infinity in that direction. By laying off the angles $-\phi_{12}/2$ and $-\phi_{13}/2$ you can find a sliderpoint S in its number one

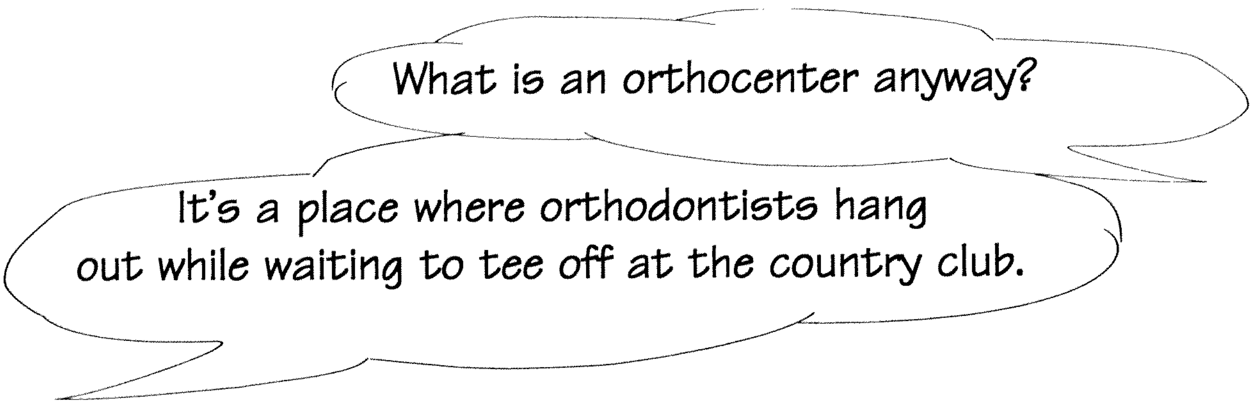


position which will be consistent with that choice of a fixed pivot. The direction of sliding will be at right angles to the line of sight to the fixed pivot.



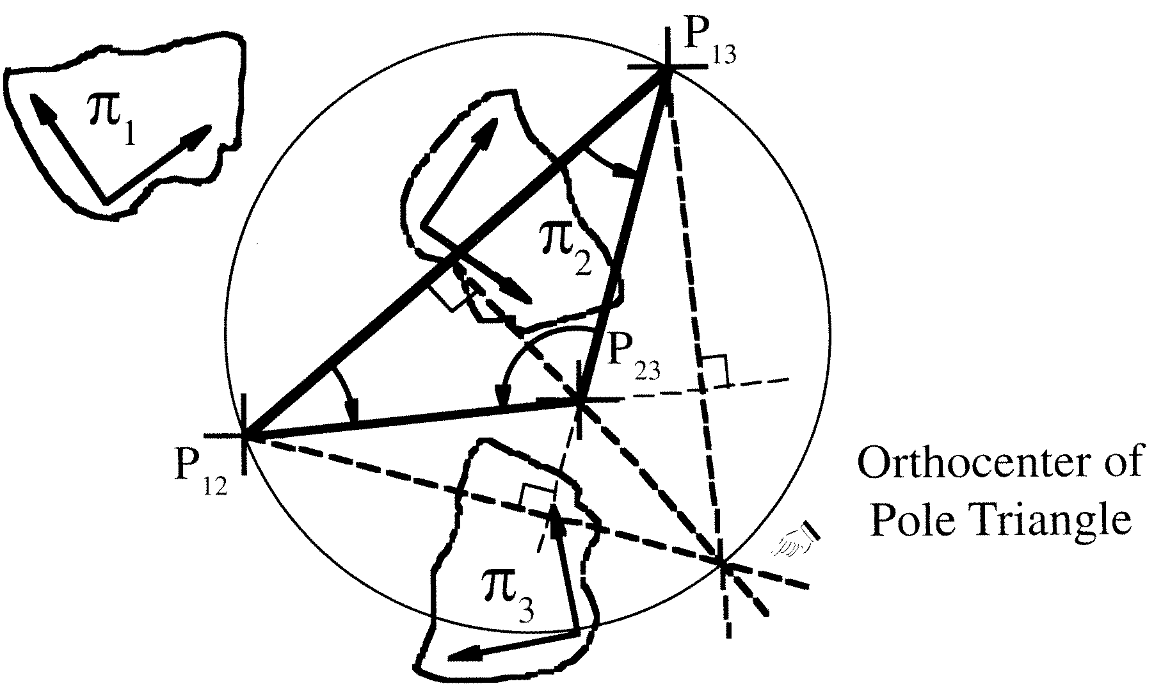
If you were to scan 360° around, looking in all possible directions for fixed pivots and plotting the corresponding moving sliderpoint locations, you would end up with a circular locus of sliderpoints.

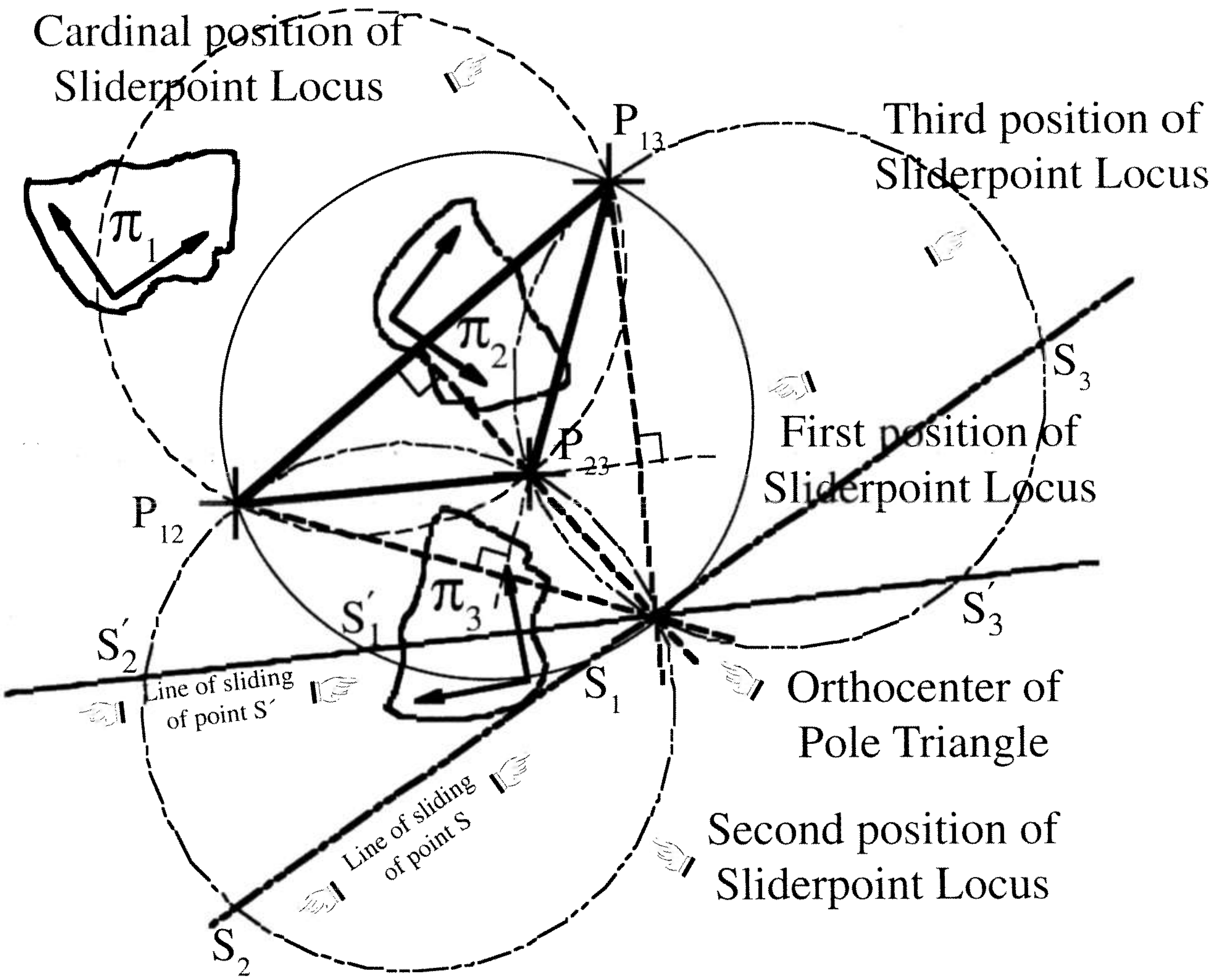
This circular sliderpoint locus is drawn on the moving body in its first position. Since it is drawn on the moving body, it travels along with the body and has an image in the second and third positions of the body as well. In the first position of the body, the sliderpoint circle passes through the orthocenter of the pole triangle and also through the poles P_{12} and P_{13}. Every point on the sliderpoint locus moves on a straight line as the body moves through the three given positions and all these lines of sliding pass through a common point which turns out to be the orthocenter of the pole triangle.



Actually, the orthocenter of a triangle is the point where the altitudes of the triangle all intersect. The image of the sliderpoint locus passes through the orthocenter in all three design positions.

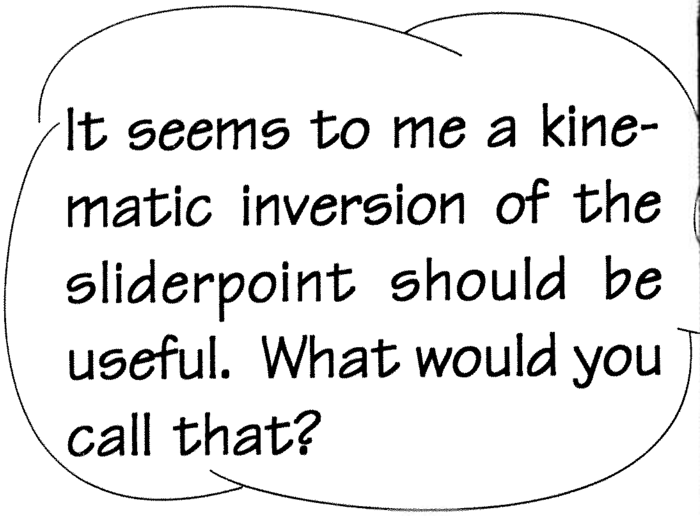
For those who have a fondness for Olympic rings, one can plot the displaced positions of the sliderpoint locus by successively reflecting the locus about the various sides of the pole triangle. In the sec-





ond position of the locus it passes through P_{12}, P_{23}, and the orthocenter. In the third position it passes through P_{13}, P_{23}, and the orthocenter.

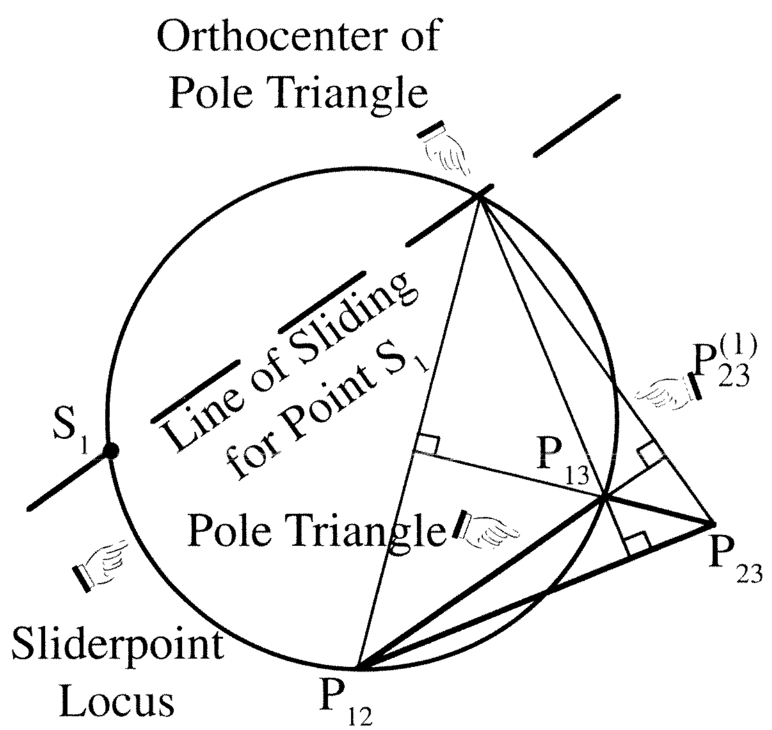
Plotting the orthocenter and these displaced circles lets you quickly visualize how any specific choice of sliderpoint will move. A sliderpoint such as S_1 in the figure above moves to S_2 and S_3 along the straight path S_1, S_2, S_3 through the orthocenter. Similarly, S´ would move from S_1´ to S_2´ to S_3´.



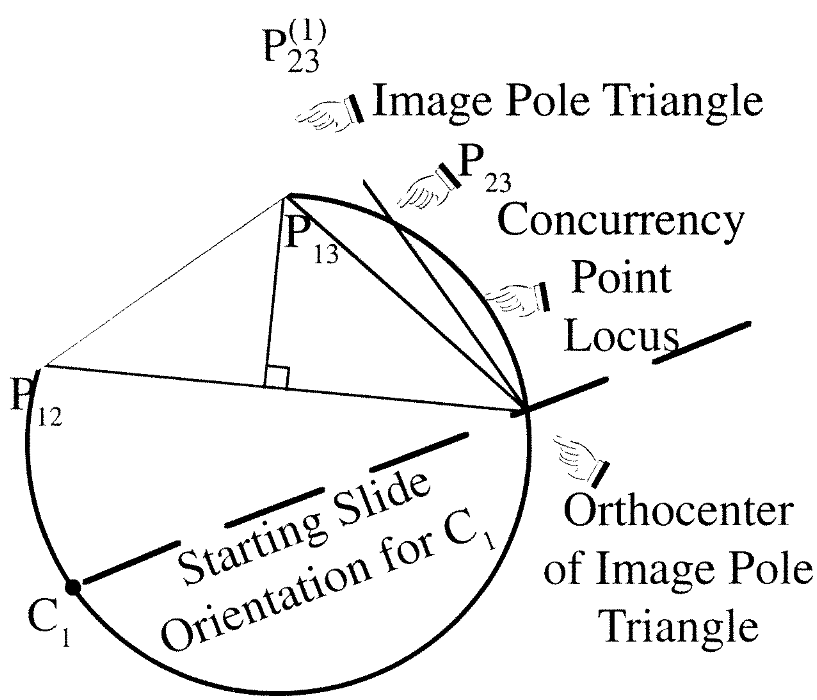
Robert Willis (1800-1875) Jacksonian Professor at Cambridge University and author of *Principles of Mechanism*. Willis felt the design of mechanisms should be attacked systematically, perhaps mathematically, to determine "all the forms and arrangements that are applicable to the desired purpose..."

I'd call it a *concurrency point*, because it is so useful you can use it like currency and take it to the bank$$. A sliderpoint is a point on the moving body that always lies on a straight line attached to the fixed body. A concurrency point is a point in the fixed body through which a straight line on the moving body always passes. Ergo, hence, to wit, forsooth, a là cart, by bus, ad nauseum, etcetera, the various positions of the straight line are always <u>concurrent</u> at the fixed point. The same way you can use a sliderpoint to design mechanisms with a slideblock hinged to the moving body and traveling on a straight guide attached to the frame, you can use a concurrency point to design mechanisms with a pivoted slideblock attached to the frame and a straight slide attached to the moving body.

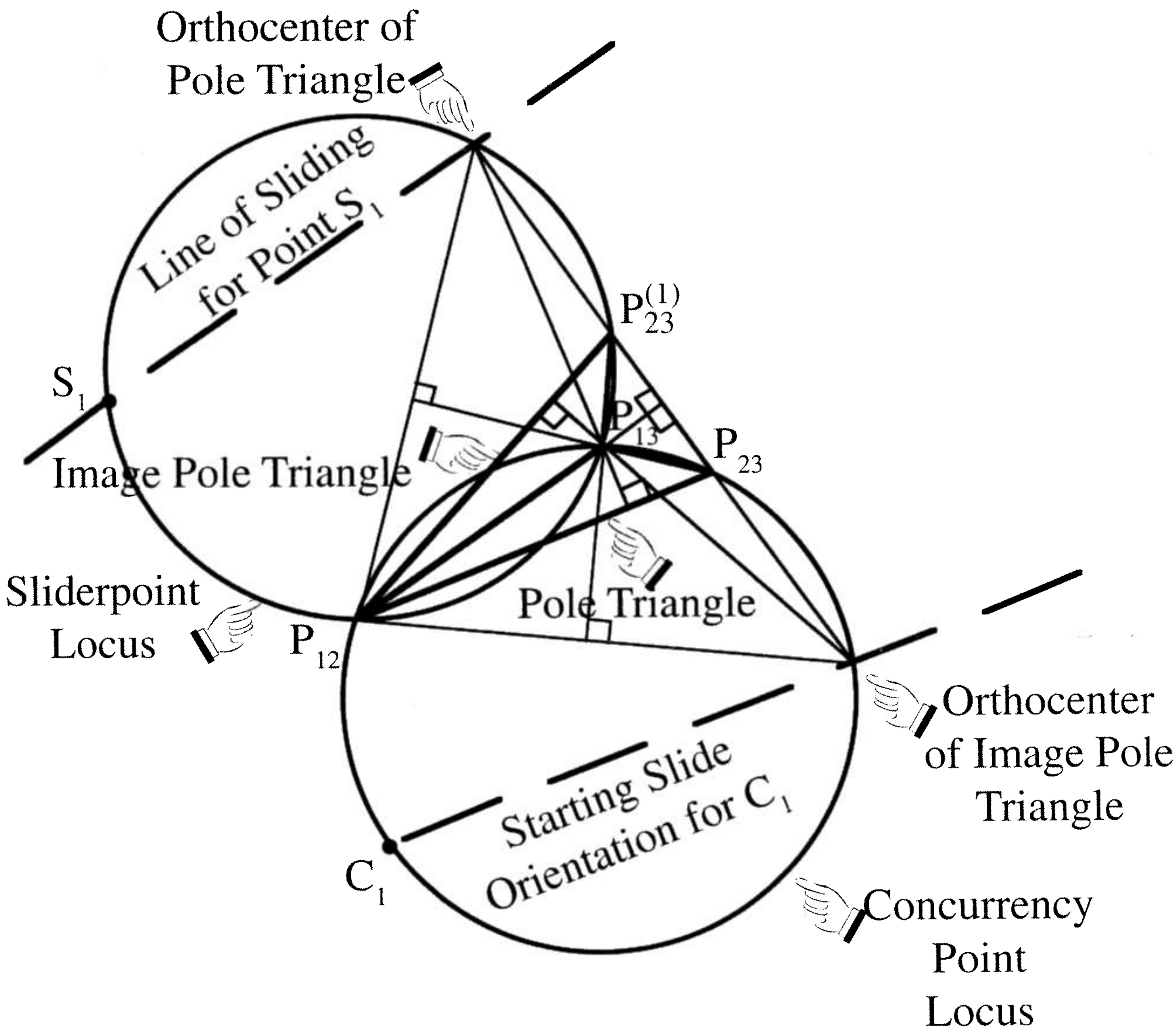
Here's what the situation looks like. In the number one position, the slider point locus passes through the poles P_{12}, P_{13}, and $P_{23}^{(1)}$. It also passes through the orthocenter of the pole triangle. Every point on that locus slides on a straight path on the frame that intersects the orthocenter of the pole triangle.



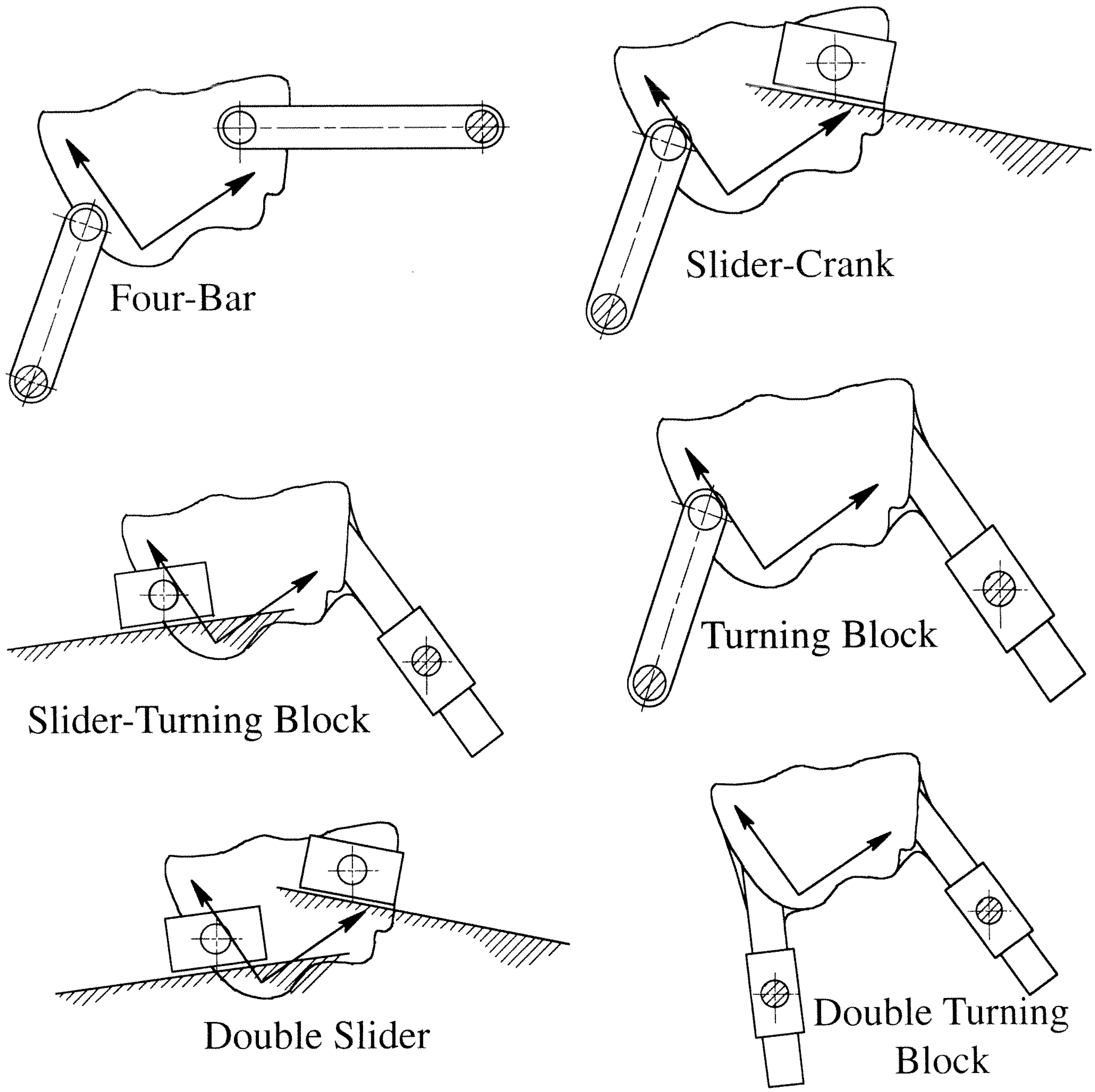
Similarly, the concurrency point locus passes through the poles P_{12}, P_{13}, and P_{23}. It also passes through the orthocenter of the image pole triangle. Every point on the concurrency point locus is a point fixed to the frame. In the first design position, there is a single straight line through each concurrency point on the frame that also passes through the orthocenter of the pole triangle. This straight line is rigidly attached to the



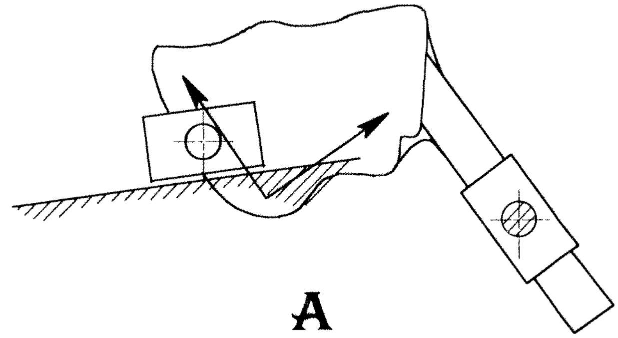
moving body. As sthe body moves through the three given positions, the line will rotate and plunge through the same point on the frame. The Sliderpoint Locus and the Concurrencypoint Locus are mirror images of one another about the line P_{12} — P_{13}, just as the pole triangle and the image pole triangle are mirror images about that same line.



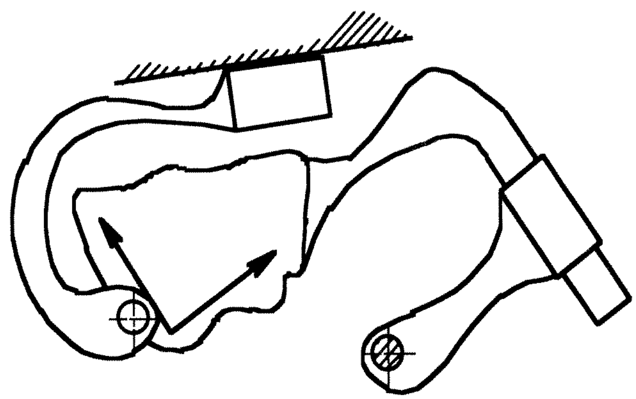
Armed with this knowledge, you can design any sort of pin or slider-jointed four-link mechanism for the purpose of guiding a body through two or three design positions. You can design garden variety four-bars, slider-cranks, double sliders, slider-turning block mechanisms, crank-turning blocks, double turning blocks, and so forth. A smattering of these devices is shown below just to whet your imagination.

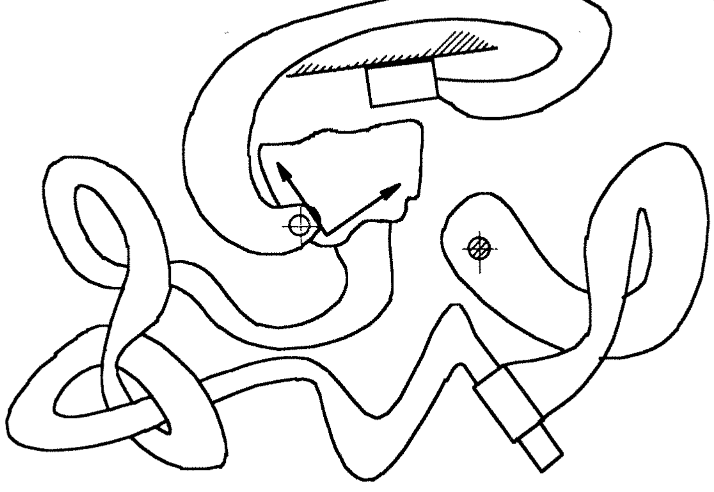


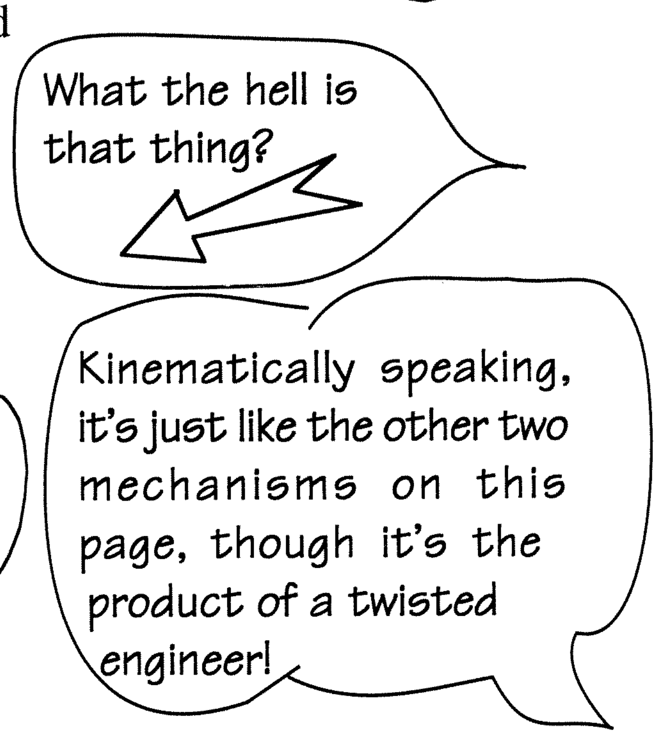
In fact, slider points and turning block (concurrency) points can be used in many neat ways that are not immediately obvious to the any but the cognoscenti. For instance, just to take this point to a ridiculous extreme, these two slider-turning block mechanisms are kinematically identical. The second one has a certain **victorian style and panache** that the top one lacks but it produces the same motion of the moving body. They may have very different frictional characteristics and your boss may fire you for designing one and not the other, but what the hell, we're talking theory here and should be above such petty and mundane considerations.

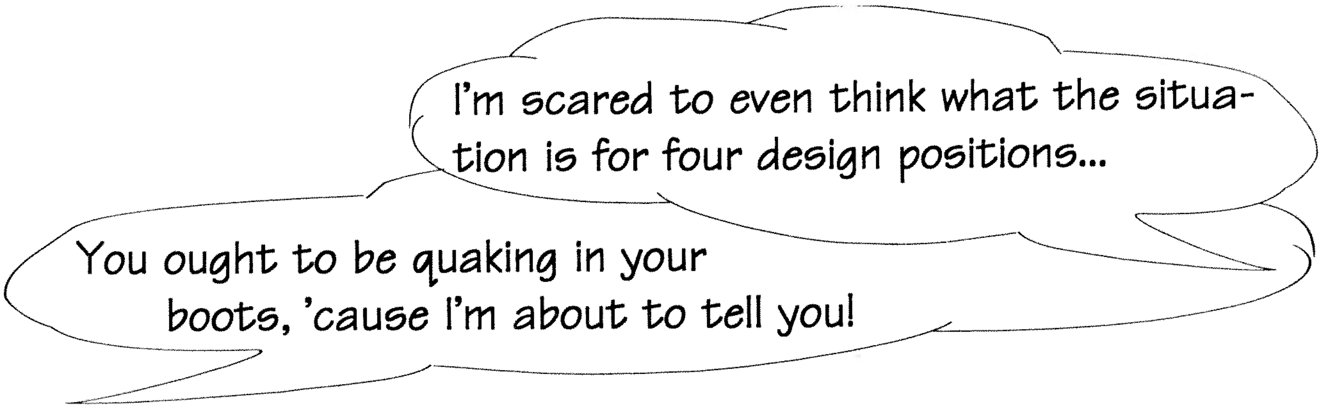


A Kinematically Matched Set of Slider-Turning Block Mechanisms





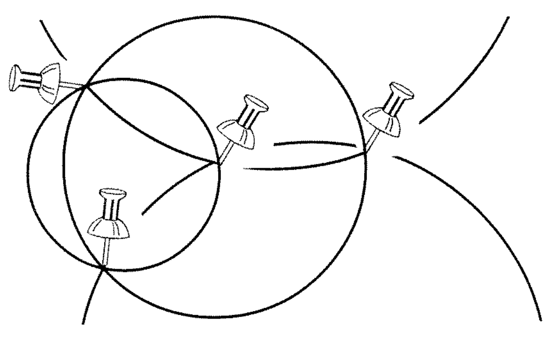




Life is cruel. As this yarn progresses and we loosely twist, turn, and embroider our way into four position synthesis we find life is crewel, crewel indeed.[†] Are we better off than we were with three positions or are we worsted?

Why is life cruel, you ask. Life is cruel because mechanism links like to swing in circles and four positions of a point you might want to use as a moving pivot seldom lie on a circle when you want them to. Get my point?

Take the following point shown in four randomly chosen locations. You can draw a circle through any three of the four positions of the point but not through all four positions at once. Damn.



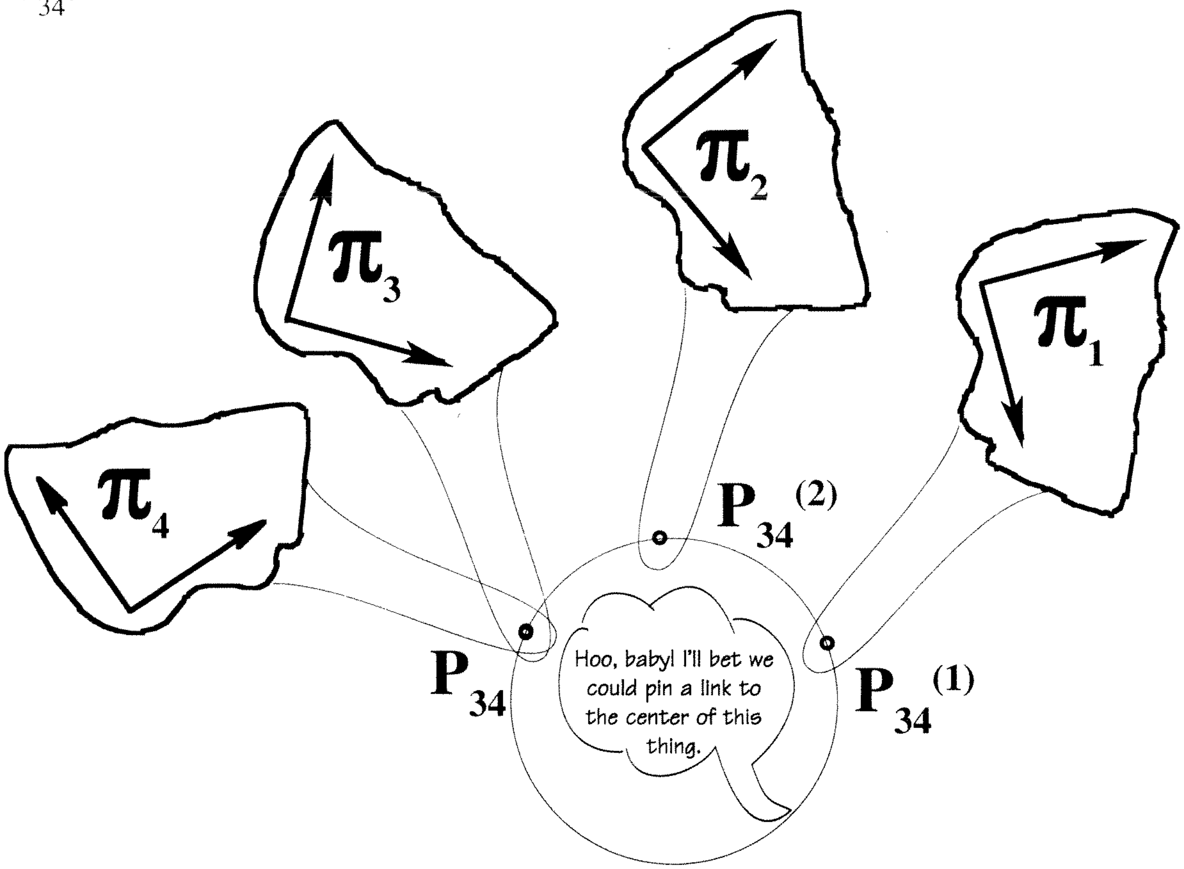
[†]Crewel: Loosely twisted worsted yarn used for fancywork and embroidery.

Just for yuckles, let's look at a *particular* point of the moving body as the body passes through any set of four given positions. Let's look at one of the poles, say P_{34}. P_{34} is common to both the moving and the fixed body when the body is in either position three or in position four. For now, concentrate on the half of point P_{34} that is attached to the moving body, rather than the coincident portion attached to the frame.

Point P_{34} is a point of the moving body that lies on a circle for *all four given positions of the body*, regardless of how oddly they might have been chosen! We can use it as a moving pivot for a linkage! To celebrate, let's give it a special name, like *"circlepoint"*.

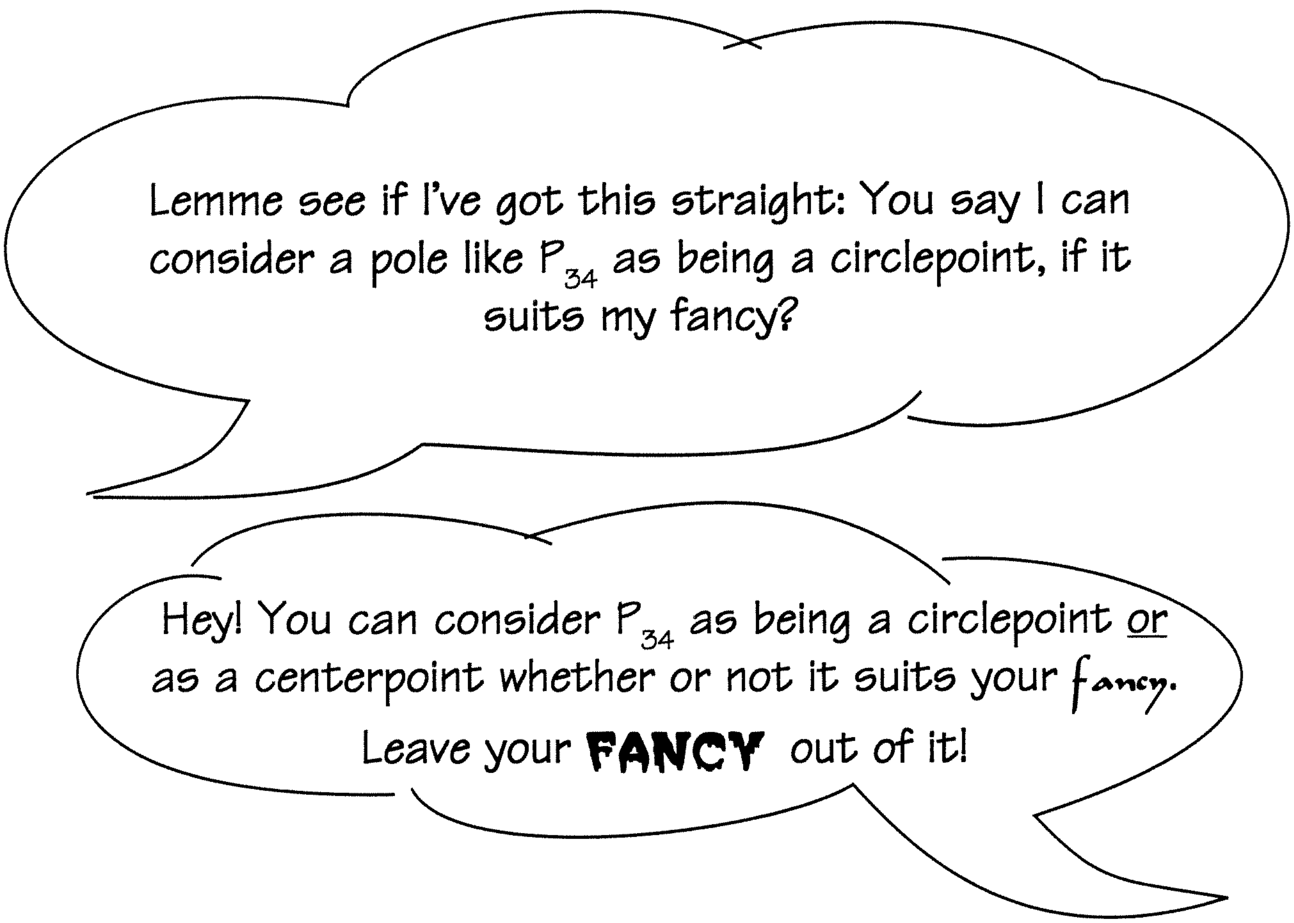
You heard me right the first time- it lies on a circle as the body goes through all four positions! Let me explain:

Point P_{34} is a point of the body that stays in the *same* spot as the body moves from position three to position four. Since it is a point of the body, it moves with the body and has an image in every other position the body goes through. Thus there is an image of the point in the first position, $P_{34}^{(1)}$ and another image of the pole in the second position, $P_{34}^{(2)}$. As the body goes through four positions, this point of the moving body only goes through three distinct positions, $P_{34}^{(1)}$, $P_{34}^{(2)}$, and P_{34}.

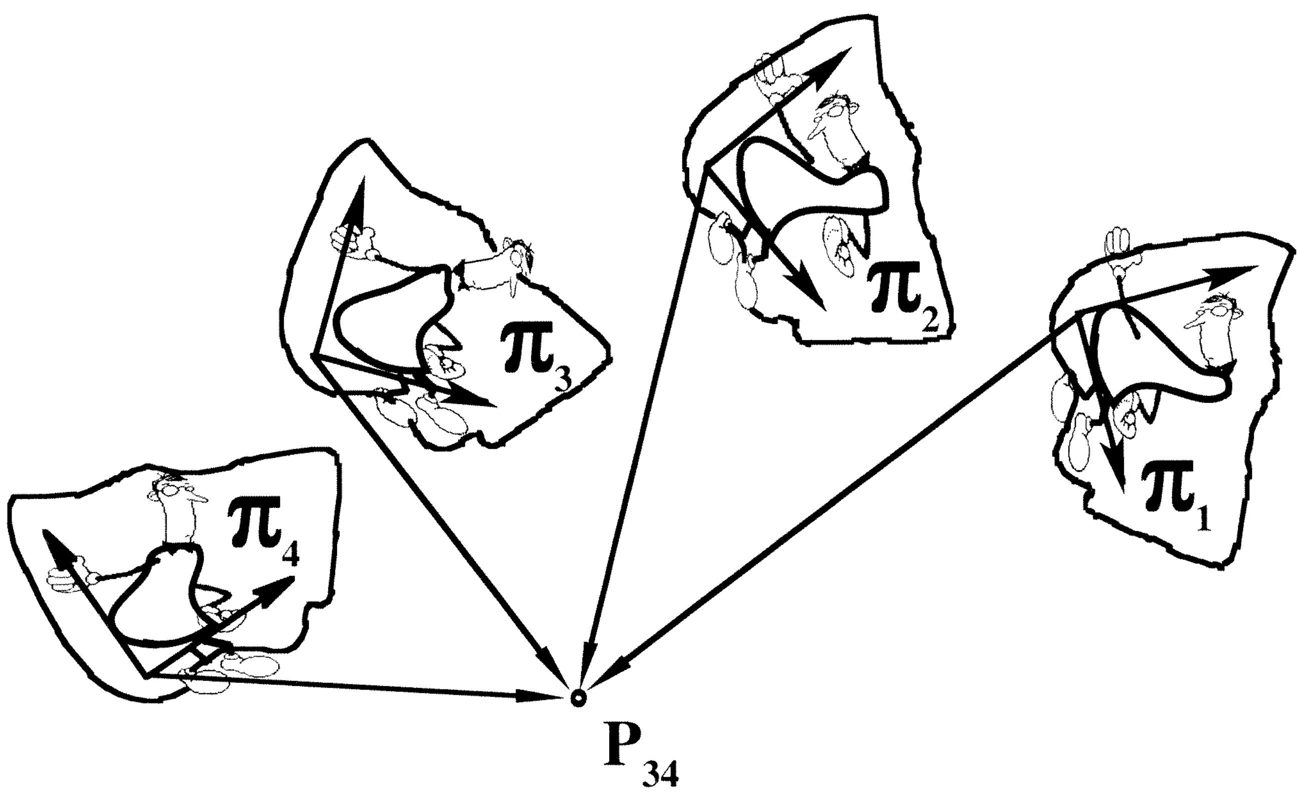


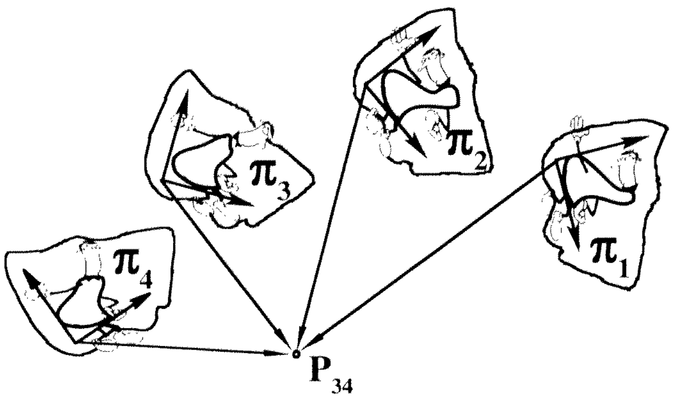
Since this point moves on a circle as the body moves through its various positions, we call it a *"circlepoint"*. That way we can sound très sophisticated. Also, since it moves on a circle, we can use it as a moving pivot for a link pinned to the frame at the center of the circle and pinned to the moving body up at the circle point. (Since the first position is our design position, we would actually lay out our mechanism by choosing $P_{34}^{(1)}$ as our moving pivot or circlepoint in its starting reference position.) As that particular link swung through just *three* positions it would guide the point of the body through *four* corresponding positions– we get one position for free, since the third and the fourth positions of that particular point happen to be coincident. The link guiding point the moving pivot from $P_{34}^{(1)}$ to $P_{34}^{(2)}$ and then on to P_{34} would dwell as the body goes from position three to position four. Since the center of the circle through $P_{34}^{(1)}$, $P_{34}^{(2)}$, and P_{34} is where we put the fixed pivot, we can really snow people with obscure technical-sounding terminology by calling the location of the fixed pivot a *"centerpoint"*.

This trick of using a pole to get one extra position for free is called *"Point Position Reduction"* and was developed by the German kinematician, Kurt Hain.

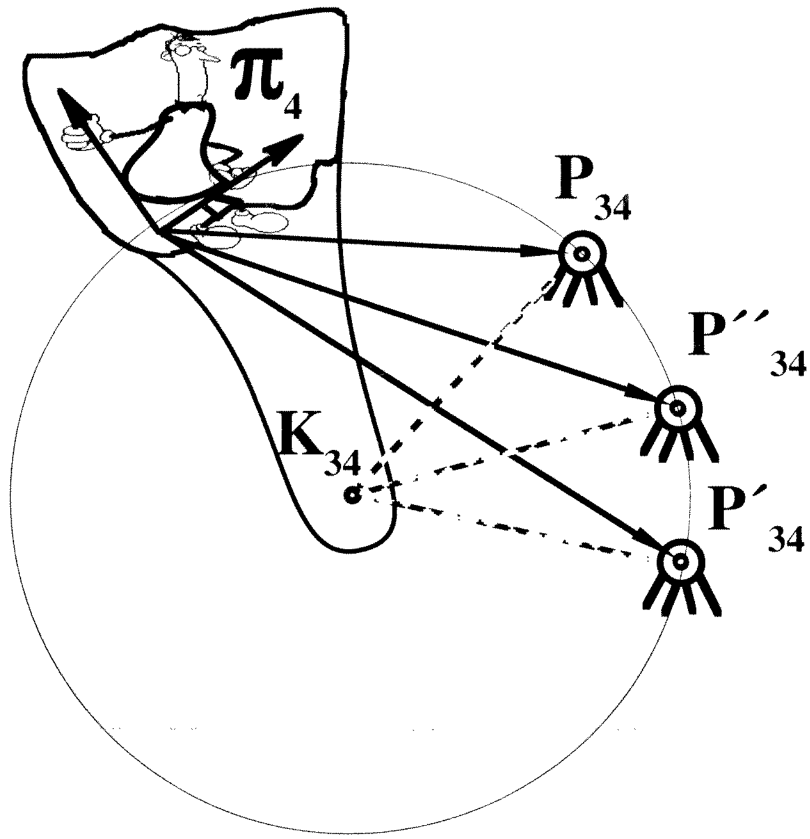


Let me run that by you again. I just showed you how you could view the half of P_{34} attached to the moving body as being a *circlepoint*, since it moves through three points that all lie on a circle as the body moves through the four given design positions. In exactly the same way, you could consider the half of P_{34} that is attached to the *frame* as being a *centerpoint* for the four positions, by just imagining how things would appear if you did a kinematic inversion and swapped the roles of the fixed and moving bodies. Imagine you were riding on the moving body, so it was your fixed frame of reference, and you were watching how point P_{34} of the frame appeared to move from your apparently stationary viewpoint.





On the left is what really happens if we stand on the fixed reference frame and watch an observer riding on the moving body as it passes through the four given design positions. Notice where the observer sees the fixed pole, P_{34} with respect to his moving coordinate system.

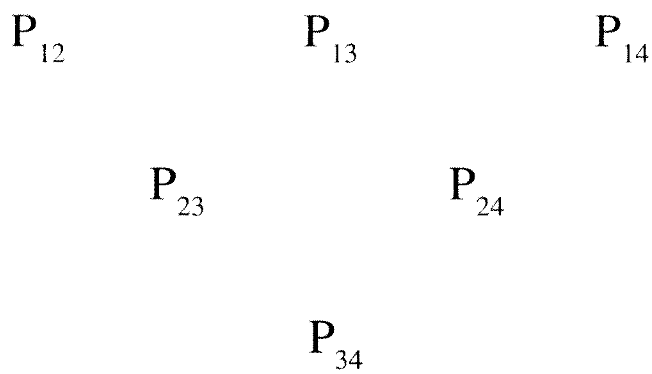


On the right is how things appear to the observer as he sits on the body when it is in position four. He is *never* aware that the body he is riding on ever moves. When the body is in position three or position four, he sees the fixed portion of the pole P_{34} in the position shown. When the body is in position two, the observer sees the fixed part of P_{34} appear to swing over to point P''_{34} and when the body is in position one, the observer sees the fixed part of P_{34} appear to swing over to point P'_{34}. From the viewpoint of our intrepid observer, the fixed part of P_{34} is apparently moving on an arc centered at a point K_{34} on the moving body. In fact, a link pinned to the moving body at point K_{34} and to the fixed body at point P_{34} would be compatible with all four positions through which the body moves, even though it only moves through three discrete positions. Thus, the pole P_{34} can be considered to be a possible *centerpoint* for the four design positions and the point K_{34} would be the matching *circlepoint* as seen when the body is in either the third or the fourth design position.

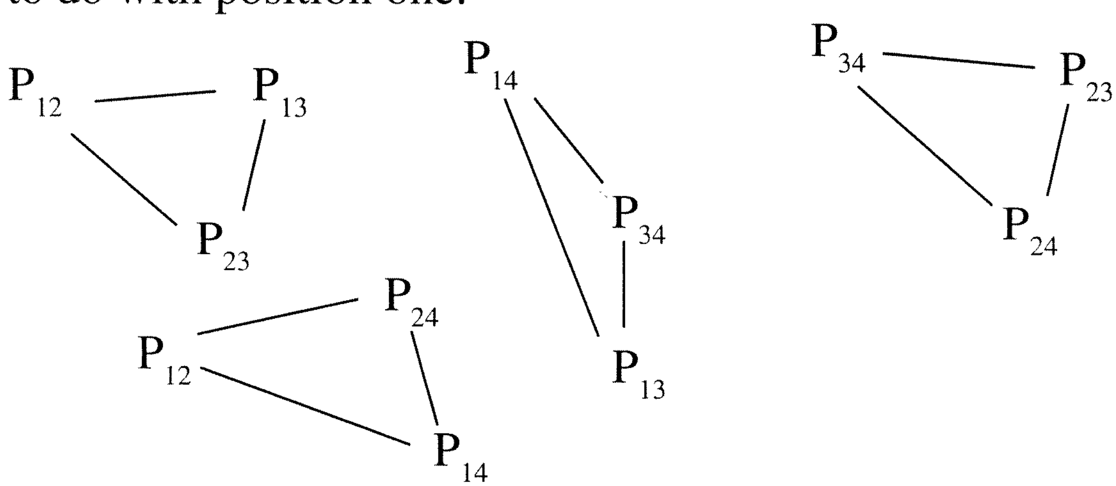
This is yet another example of Hain's method of *Point-Position Reduction*. You can use variations on this theme with any of the poles or image poles. Just keep your wits about you and think whether you want to work with the portion of the pole attached to the fixed body or the part on the moving body. There are a zillion ways to carry out the process of kinematic inversion and to finagle this technique. There are ten zillion ways to do it wrong.

Be sure you do all your graphical constructions in a consistent design position. Tracing paper can be a convenient way to handle the job of transfering points from one design position back to a consistent reference position (if you don't have the luxury of a nice computer graphics system to do the job). The nice thing about tracing paper is you can ball it up and throw it at something or somebody if things aren't working out the way you hoped they would.

Four positions of a rigid body determine the plot of a Murder She Wrote episode. Alternatively, four positions of a rigid link determine the following six poles:



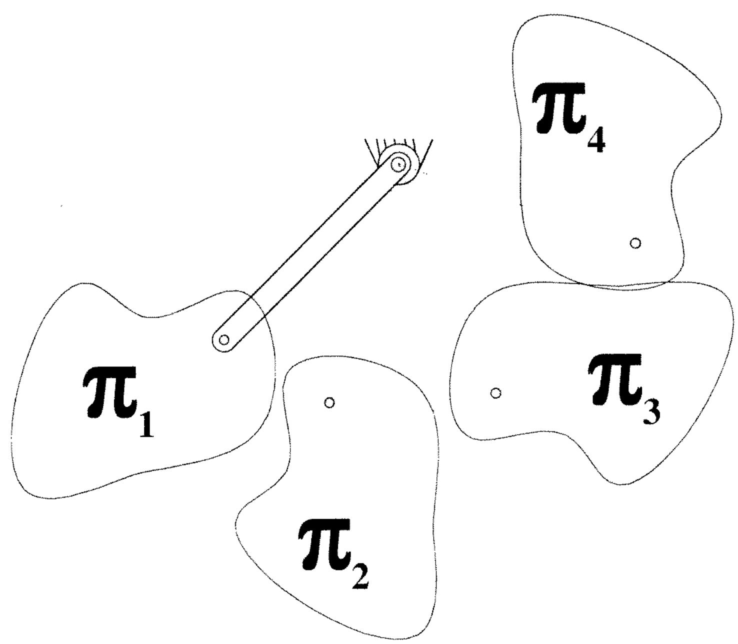
As you may have feared, there are four pole triangles you could form from those six poles. (Throw away any one of the four positions and you can form a pole triangle from the three remaining positions.) You can form a triangle using poles P_{12} P_{13} P_{23} (leaving out any poles or other information having to do with position four). Similarly, you can leave out position three and form a triangle using poles P_{12} P_{14} P_{24}. Skipping position two gives pole triangle P_{14} P_{13} P_{34}. Last but not least, pole triangle P_{23} P_{24} P_{34} incorporates no information having to do with position one.



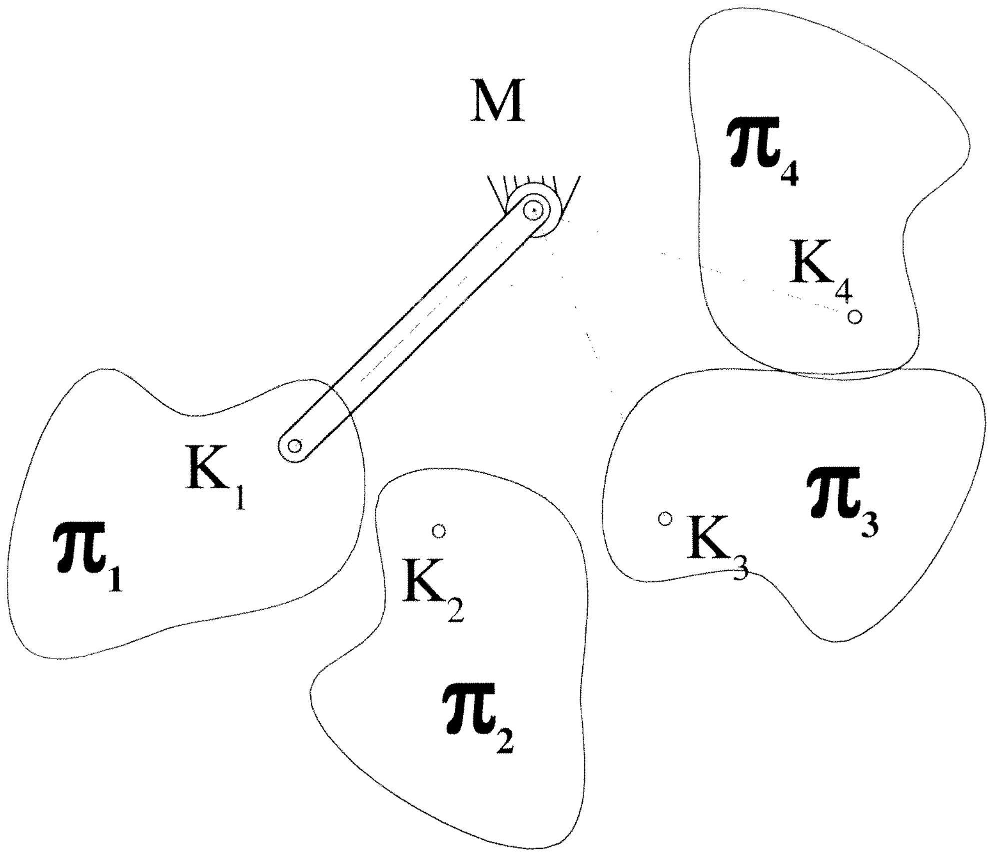
Whatever shape they may turn out to be, all four of these pole triangles are *Pole Triangles* so they behave like ***Pole Triangles Have Previously Been Shown to Behave!*** Howzat, you ask. Like a blankety blank pole triangle is howzat! Therefor, they have the well-known properties you learned earlier: their interior angles are half the corresponding angles of rotation, etcetera, etcetera, etcetera, etcetera, etcetera, etcetera.

Enough about poles, pole triangles and their properties for now. Let's get back to the task of designing a link for a mechanism to guide a body through four given positions. How the hell are we gonna do that??

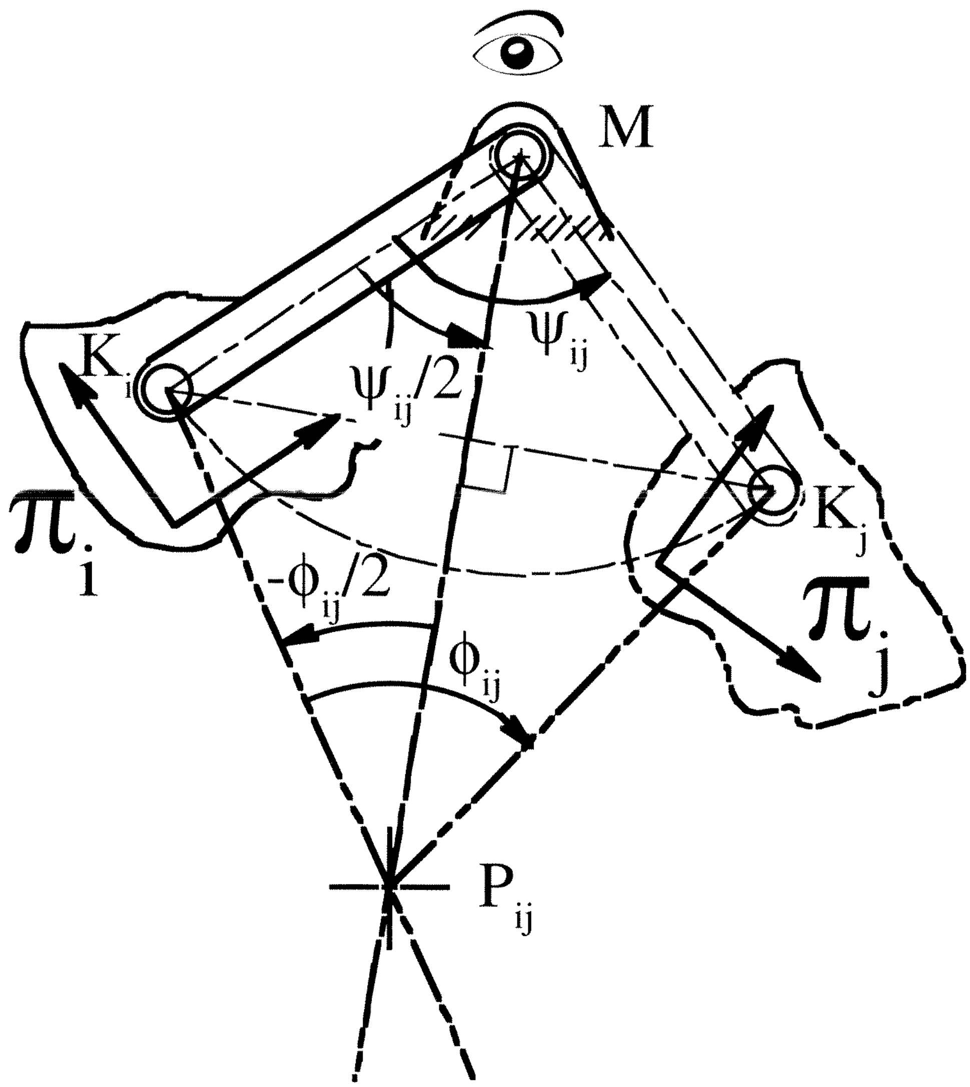
Suppose we somehow had actually *found* a link that could guide a point of our moving body through it's four given design positions. (Maybe we used a Ouigi board to do it. For now, let's not worry about how we got this link. Let's just study it's properties a bit and see what we can learn from it.)



This link has a fixed pivot (or centerpoint) we'll call M and a moving pivot (or circlepoint) we'll call K. When the body goes through the four given positions, the circlepoint goes from K_1 to K_2 to K_3 to K_4.

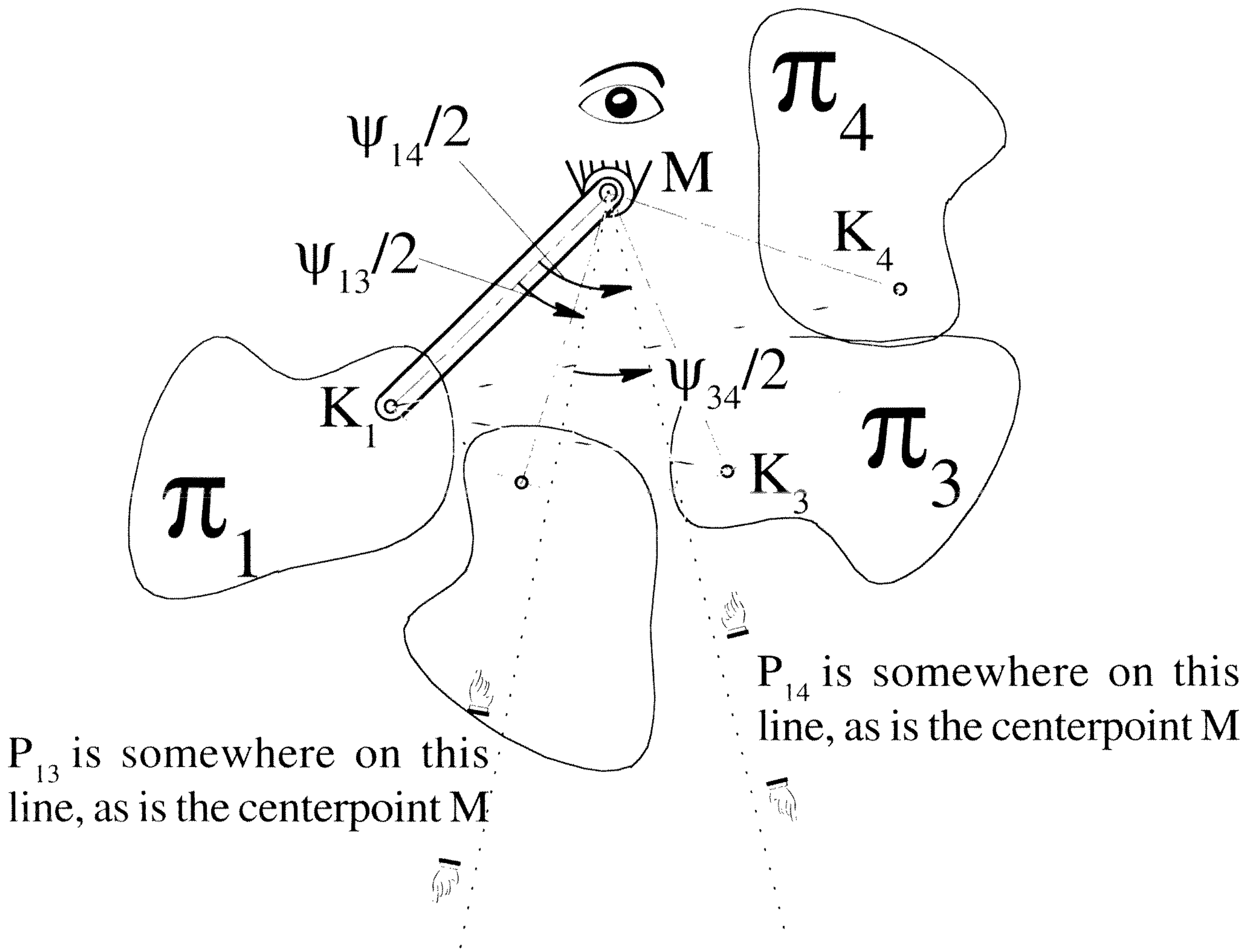


Let's look at what happens as the link swings through any two of the design positions. As the link swings from the i^{th} position to the j^{th} position, it swings through some angle ψ_{ij} while the body is rotating about the pole P_{ij} by the specified angle φ_{ij}. The angle φ_{ij} is known from the specified design data and has *nothing* to do with the points K or M. On the other hand, notice that the angle of swing of the *link* depends entirely on the particular link we are studying. (Had we picked a different circlepoint K and centerpoint M we would have had a completely different swing angle.)



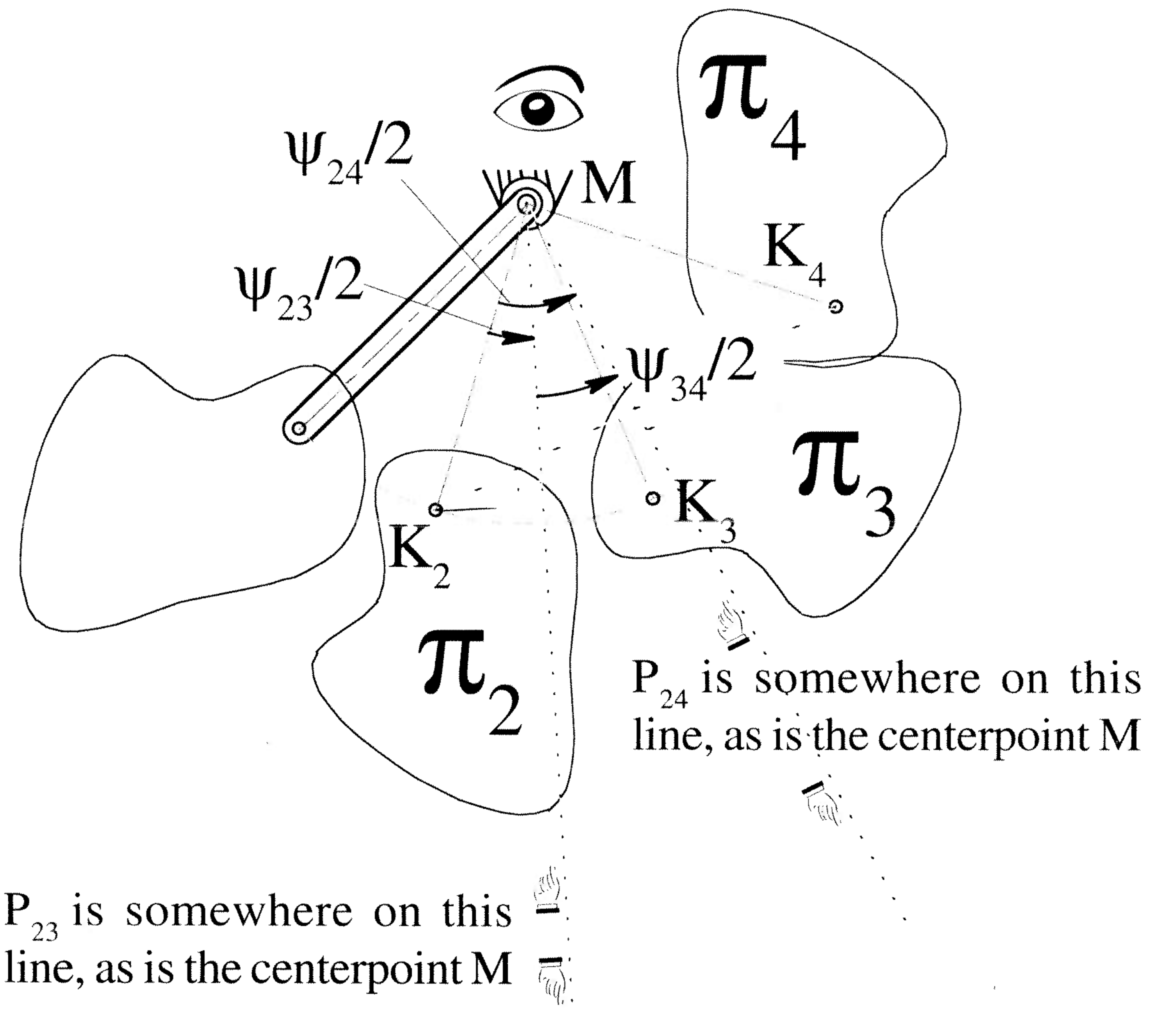
If we put our eyeball on the point M (figuratively speaking— don't actually rip out any body parts) we see that the pole P_{ij} is located on the perpendicular bisector to K_i — K_j and that the angle from the i^{th} position of the link to the perpendicular bisector passing through the pole P_{ij} is ψ_{ij} / 2.

In particular, let's watch what happens as the link swings between positions 1 and 4 and then between positions 1 and 3:

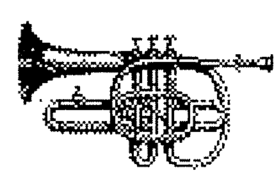


We see from the figure above that the angle from K_1 to M to the perpendicular bisector to K_1 — K_4 is $\psi_{14} / 2$. Similarly, the angle from K_1 to M to the perpendicular bisector to K_1 — K_3 is $\psi_{13} / 2$. If we subtract one of these angles from the other, we see that the angle from P_{13} to M to P_{14} is some angle $\psi_{34} / 2$. (The particular value of this angle depends on the particular choice of point M.)

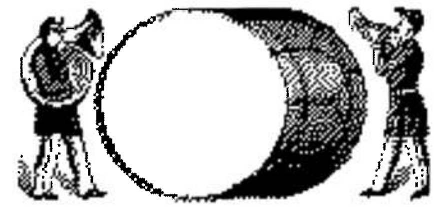
Suppose we look at positions 2, 3 and 4 instead. We'll see a similar situation.



Look what we've done! From this figure we see that the angle from K_2 to M to the perpendicular bisector to K_2 — K_4 is ψ_{24} / 2. Similarly, the angle from K_2 to M to the perpendicular bisector to K_2 — K_3 is ψ_{23} / 2. As before, if we subtract one of these angles from the other, we see that the angle from P_{23} to M to P_{24} is *the same* angle ψ_{34} / 2 that we found before. **Ta dah!**



Why "ta dah", you ask? **I'll tell you why *ta dah*.** In fact, this calls for ***ta dah*** followed by a drum roll!



What we have discovered here is a way to identify valid centerpoints for four arbitrarily specified positions of a moving body, should we ever happen to stumble upon one of them.

In general, three positions of a point define a circle and four random positions of a point *won't* lie on a circle except in rare cases. In four position synthesis, you are faced with a situation where you are given four desired positions through which you want to guide a body. Any old point you might select in that body will lie on a circle for three of the four positions but will fall off that circle in the remaining position.

Links of pin jointed linkages kind of like to swing on circles. I don't know why— it's just the way they are. In a non-Euclidean universe maybe they'd swing on Lobachevskian donuts but to me the idea Gauss against the grain. But I digress...

The question is, just because *any* old random point you pick in a body moving through four positions doesn't lie on a circle in all four corresponding positions doesn't mean there aren't *some* points in that body that *will* lie on a circle for all four of their corresponding positions. The trick is to be smart enough to be able to recognize them when or if you ever see one.

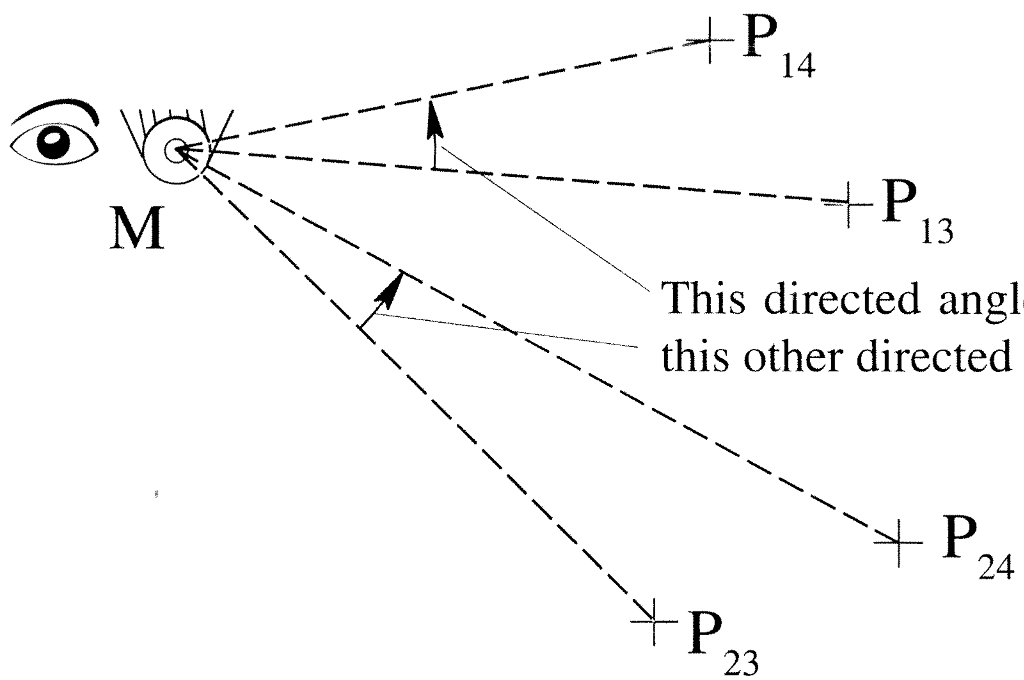
Well, maybe you wouldn't recognize a four position circlepoint if you happened to stumble across it but I just showed you how to recognize the *centerpoint* that is the center of the circle that that *circlepoint* was traversing! Once you have the center of a circle it kind of makes it easier to find the circle itself. (Hence and forsooth, ***ta dah!***)

Let me recap: If, as we wander around on the fixed reference body, we ever happen to step on a point from which we see poles P_{13} and P_{14} as being under the *same* angle as that between the poles P_{23} and P_{24} we should stop and scream **"Aha! I am standing on a centerpoint!"** Pull out your magic marker and mark the spot before you lose it, cause there aren't very many places from which you can make that claim. (Well, it turns out there are an *infinite* number of places from which you can make that claim, but that is still just a drop in the bucket, since there are a doubly infinite number of places from which you *can't* make that claim.)

While yer doin' that I think I'll have a little nightcap myself!

Coleman Lantern used by modern day searchers looking for an honest man, special prosecuters looking for an honest politician, political fund raisers looking for a buck under a rock, and kinematicians looking for a centerpoint out of tens of zillions of non-centerpoints.

Notice we are dealing with two pairs of poles— a pair with a 1 in common and another pair with a 2 in common. Both these pairs have 3 and 4 as the non-common subscripts. If, when we look from the 3 to the 4 pole in the set with the common 1 subscript we see the same angle (both in magnitude and direction) that we see as we look from the 3 to the 4 pole in the set with the common 2 subscript, then we know we are standing on a centerpoint for the four positions for which these are poles.



In fact, it turns out that there is even a more general way we can ascertain we are standing on a centerpoint. Shown above is just one particular case.

Pairs of Poles that *don't* have a common subscript are called *Opposite Pole Pairs*.‡

‡ Pretty Perplexing & Peculiar Properties Paradoxically Predominate Practical Paradigms Pertaining to Preponderance of Pole Pairs. Professor Participates as Paraclete & Polyhistor, Perhaps Primoprime Popinjay Pundit Preventing Paltry Poultry Problems, PoleOPhobia, Poletrusions, & Poleidonia.